The complex sine-Gordon model on a half line

Tzamtzis, Georgios

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Georgios Tzamtzis

Centre for Particle Theory

Department of Mathematical Sciences

A Thesis presented for the degree of Doctor of Philosophy

at the University of Durham

England

March 2003

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Georgios Tzamtzis

Ph.D. Thesis, March 2003

Abstract

In this thesis, we study the complex sine-Gordon model on a half line. The model in the bulk is an integrable (1+1) dimensional field theory which is $U(1)$ gauge invariant and comprises a generalisation of the sine-Gordon theory. It accepts soliton and breather solutions. By introducing suitably selected boundary conditions we may consider the model on a half line. Through such conditions the model can be shown to remain integrable and various aspects of the boundary theory can be examined.

The first chapter serves as a brief introduction to some basic concepts of integrability and soliton solutions. As an example of an integrable system with soliton solutions, the sine-Gordon model is presented both in the bulk and on a half line. These results will serve as a useful guide for the model at hand. The introduction finishes with a brief overview of the two methods that will be used on the fourth chapter in order to obtain the quantum spectrum of the boundary complex sine-Gordon model.

In the second chapter the model is properly introduced along with a brief literature review. Different realisations of the model and their connexions are discussed. The vacuum of the theory is investigated. Soliton solutions are given and a discussion on the existence of breathers follows. Finally the collapse of breather solutions to single solitons is demonstrated and the chapter concludes with a different approach to the breather problem.

In the third chapter, we construct the lowest conserved currents and through them we find suitable boundary conditions that allow for their conservation in the presence of a boundary. The boundary term is added to the Lagrangian and the vacuum is reexamined in the half line case. The reflection process of solitons from the boundary is studied and the time-delay is calculated. Finally we address the existence of boundary-bound states.

In the fourth chapter we study the quantum complex sine-Gordon model. We

begin with a brief overview of the theory in the bulk where the semi-classical spectrum and an exact S-matrix are presented. Following that we use the stationary phase method to derive the semi-classical spectrum of boundary bound states. The bootstrap method is used as an alternative approach to obtain the same spectrum. The results are discussed and compared.

The final chapter consists of a general discussion on open questions and problems of the model, and some proposals for further research.

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Declaration

This thesis summarises the research work carried out by the author from May 1998 to March 2003 at the Centre of Particle Theory at the Department of Mathematical Sciences of the University of Durham, England. No part of this thesis has been submitted for any other degree in this or any other University.

Chapter 1 and sections 2.1 up to 2.5 of chapter 2 serve as introductory material for which no claim of originality is made. The remaining of chapter 2 and chapters 3 and 4 are believed to be original work, unless stated otherwise. The bulk of the original research was carried out in collaboration with my supervisor Dr. Peter Bowcock and appears in the papers [1, 2].

Acknowledgements

I wish to take this opportunity to thank all the people who have helped me all these years and contributed to the completion of this work.

First of all I would like to express my deepest gratitude to my supervisor, Dr. Peter Bowcock. His patience, encouragement and guidance during the preparation of this work have been invaluable.

I would also like to thank Dr. Patrick Dorey for many useful discussions, Prof. D. Ghikas for help and guidance, and everyone in the Department of Mathematical Sciences who I had the pleasure to work with. Special thanks to Sharry for all his assistance and our pleasant discussions during all these years.

My thanks go also to all the people who made my stay in Durham a very enjoyable one. With no order of preference : Leo, Matthias, Apostolis, Ana, Kostas, Babis, Yannis M., Christos C., Christos S. (a great coffee partner), Dimitris S. , Andreas A, Eleanna, Thani (for our endless discussions), Pantelis (for keeping in touch) and George S. (for being there). Special thanks to Dawn W. for her patience and encouragement all these years.

In addition I would like to thank my colleague, co-adventurer but above all good friend for over a decade, Yannis Floratos.

Most crucially I would like to thank my parents and my sister Maria. Their love, support and encouragement have always been overwhelming and no words of gratitude can ever be enough.

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Chapter 1

Introduction

1.1 Introduction and Outline

Modeling the physical world is one of the most challenging and rewarding aspects of human intellect. It has become an inseparable part of scientific progress and has laid the foundations for the development of new technologies. From simple everyday mechanics to the unification of fundamental forces, the underlying idea remains the same; to find a compact, coherent and consistent mathematical description, within which we can explain physical observations.

This has never been an easy task. The physical world is incredibly rich in structure and there are a vast number of physical phenomena that have to be accounted for within the framework of a mathematical model. Even with a 'simple' grand unifying theory of fundamental forces behind it, the universe exhibits an overwhelming variety of phenomena that have to naturally arise from the proposed model. Moreover, as the uncertainty principle and the constancy of the speed of light have demonstrated, basic ideas which are essential in the modeling process may be counterintuitive. Quantum mechanics and relativity are prime examples of how fundamentally different new models may be from their predecessors.

It is clear from the above that it is especially difficult to construct a consistent mathematical description of a physical system. Also correctly analysing and solving the proposed model itself proves an equally rigorous task. It is therefore wise to start with models that can be easily solved but nevertheless exhibit a rich and complicated behaviour. Once these models have been studied thoroughly and understood completely, they can provide useful insight to more complicated theories. It is in this spirit that many researchers choose to study simple models. This effort has not only led to incredibly interesting results, but in the process they have managed to develop new techniques to tackle several difficulties that have arisen. Today, new objects like solitons, vortices and instantons are parts of field theory, while the inverse scattering transform, the Hirota method and the Bäcklund transformation are widely used techniques for generating solutions.

The majority of such studies are focused on field theory models described by non-linear partial differential equations in one space and one time (1+1) dimension. The choice of dimensions is made in order to simplify an already difficult problem but also because through symmetries, realistic higher dimensional problems may be reduced to 1+1 dimensions. Generalisation to higher dimensions can also be made, although this is rarely a straightforward task.

The non-linearity provides the model with a complex interacting behaviour which is not only interesting from a mathematical viewpoint but also reflects the interacting nature of most physical systems. Nature seems to favour subtle non-linear behaviour to that of easily solvable but trivial linear systems. However, non-linearity ensures that finding explicit solutions to these equations is extremely awkward. A perturbative approach is probably the only way to go, but even this is not always applicable for instance if the coupling constant is strong as is the case in QCD. Nevertheless it is possible to find in nature complex non-linear behaviour which can be approached through models which are (or nearly are) completely solvable. This means that for these models exact analytic solutions can be written down. These models are known as integrable models and enjoy a special status due to their remarkable features.

This thesis examines one such integrable model, the complex sine-Gordon model. It is a (1+1) dimensions field theory which appeared in the literature almost 30 years ago and belongs to a family of generalisations of the also integrable sine-Gordon model. The model enjoys all the nice features of complete integrability, while at the same time introduces an internal $U(1)$ degree of freedom. It admits both topological and non-topological soliton solutions as well as breather solutions which collapse to the corresponding solutions of the sine-Gordon model in the chargeless limit.

The outline of the thesis is as follows. In this introduction some key-point elements of integrability are presented. A small discussion about soliton solutions and their basic features follows and the introduction concludes with a brief description of the sine-Gordon model and some important results which will be needed for a qualitative comparison with the model at hand. The final part of this chapter introduces the two different methods that will be used in the fourth chapter when we address the quantisation of the model.

In the second chapter, we present the model and continue with a brief literature review on the complex sine-Gordon model. Different aspects of the model are presented and a suitable vacuum is chosen in terms of the Wess-Zumino-Witten interpretation of the model. We examine classical solutions in detail, beginning with the single soliton and continuing with the two-soliton solution derived from the Bäcklund transformation for the complex sine-Gordon model. We discuss the existence of breather solutions and introduce a consistent way to create them in the absence of direct confirmation through the equations of motion. The relation of these solutions to the other multi-soliton solutions is also briefly discussed.

In the third chapter, we introduce a boundary term to the Lagrangian and consider the model on a half line. We use the zero curvature representation to find the infinitely many conserved quantities and through them, boundary conditions that preserve the integrability of the model. Once such conditions have been found, a reexamination over the vacuum structure is made to establish whether the introduction of the boundary demands a new non-trivial vacuum state. We study the scattering of solitons off the boundary through the method of images and calculate the time delay induced by this process. Finally we explore the spectrum of boundary bound states beginning with the static single soliton solution which is shown to satisfy the boundary conditions. The chapter concludes with the necessary conditions for the existence of boundary bound breathers.

In the fourth chapter we examine the quantum case of the complex sine-Gordon equation on a half line. At first we review the results in the literature about the quantum theory in the bulk and we continue with the quantisation of the boundary problem. We use the semi-classical methods introduced by Dashen, Hasslacher and Neveu [3, 4] and refined by Corrigan and Delius in [5] to obtain the spectrum of states up to the second order corrections in the expansion of the coupling constant. Alternatively, we use the bootstrap method of Ghoshal and Zamolodchikov [6] to derive the quantum spectrum. Finally a direct comparison of the results is made in order to establish exact relations between the introduced parameters of the two methods.

In the final chapter we present a brief overview of the most important results and discuss further research possibilities.

In the appendix the theory of optical pulse propagation is briefly presented. We begin with a small discussion about the field theory description of the Maxwell-Bloch theory in terms of the sine-Gordon model. The reformulation of the theory in terms of the complex sine-Gordon model and the benefits of this generalised version of the sine-Gordon theory are discussed. Once again we use the matrix potential formalism to demonstrate integrability and in contrast with the second chapter, a different gauge is chosen for a more realistic physical description. We demonstrate that the infinite number of conserved charges may again be constructed with a suitable choice of boundary conditions.

1.2 Integrable models

It is a rather difficult task to determine whether a model is integrable. The main reason is that there is no clear definition of what integrability is. To approach and define integrability, one can choose many a path ranging from algebraic geometry to conservation laws. Since most integrable systems possess a Hamiltonian structure, we choose to examine integrability based on elements of this formalism. In contrast with a strict mathematical approach, through the Hamiltonian formalism we can identify elements of integrability with properties of the physical system. We begin with a finite-dimensional Hamiltonian model before we proceed to a field theory.

A Hamiltonian model of $2n$ independent variables is completely integrable if there exist n functions $Q_i(q,p)$ which are independent that satisfy the following relations

$$\{\mathcal{H}, Q_i(q,p)\} = 0 \quad \text{and} \quad \{Q_i(q,p), Q_j(q,p)\} = 0 \quad , \quad i = 1..n$$

where $\mathcal{H}$ is the Hamiltonian of the system. The Q_i functions are called commuting integrals of motion, or conserved quantities. The Hamiltonian $\mathcal{H}$ depends on the generalised variables (q,p) only through the conserved quantities, i.e. $\mathcal{H} = \mathcal{H}(Q_i)$.

A field theory model is completely integrable when it is realised as an infinite-dimensional extension of a finite Hamiltonian system. As now n goes to infinity so does the number of integrals of motion. This infinite number of conservation laws implies an infinite number of underlying symmetries which are associated with an infinite-dimensional algebra.

Although difficult, it is possible to make the following change of variables

$$(q,p) \to (\phi, I) \ ,$$

so that the following relations are true

$$\{\phi_i, I_k\} = \omega_{ik}(Q) \quad \text{and} \quad \{\mathcal{H}, \phi_i\} = \omega_i(Q) \ .$$

The new variables I and ϕ_i are called action-angle variables. Through this transformation the Hamiltonian structure is maintained and the system becomes completely separable as the equation of motion reduce to the infinite set of equations

$$\begin{aligned} \frac{d}{dt} I = \{\mathcal{H}, I\} &= 0 \ , \\ \frac{d}{dt} \phi = \{\mathcal{H}, \phi\} &= \omega(I) \ , \end{aligned}$$

which can be easily solved. The canonical transformation used to change the system into the action-angle variables is known as the inverse scattering transform. It is a non-linear version of the Fourier transform in linear systems and may be used to generate solutions when time independent solution are already known. This method was originally devised and applied to the Kortweg de-Vries equation by Gardner, Green, Kruskal and Miura in a series of papers [7]. Later the method was also used successfully in a number of different models thus establishing it as a consistent procedure for solving non-linear problems. The idea behind this is through the scattering data of an already known solution to reconstruct the time-dependent solution. Schematically one has

$$\begin{tikzcd} g(x,0) \arrow[r, "\text{Scattering Transform}"] & S_0 \arrow[d, "\text{Time Evolution}"] \\ g(x,t) & S_t \arrow[l, "\text{Inverse Scattering Transform}"'] \end{tikzcd}$$

One begins with a time independent solution $g(x,0)$ and performs a scattering transform to get the scattering data S_0. The linear character of the latter makes their time evolution an easy task. Finally one performs the inverse scattering transform on the time evolved data S_t to recover a time-dependent solution to the non-linear equation. It is an exact copy of the method used in linear systems for finding, solutions, with the scattering transformation being replaced by the Fourier transform.

A significant factor in the development of the inverse scattering method was the computational power that became available at that period. However, the numerical calculations only helped establish a firm basis for turning the inverse scattering transform into a consistent and generally applicable method.

The modern formulation of the inverse scattering method, is the Lax pair approach. It was developed by Lax in [8] in a treatment of the integrable Kortweg de-Vries (KdV) equation. Using the results of Gardner, Green, Kruskal and Miura, Lax managed to rewrite the theory as a linear spectral problem, expressing the equation of motion as an equation between two suitably chosen linear operators called the Lax pair. The Lax method can be applied also to other integrable systems. As it turns out, all integrable systems possess a Lax pair. Conversely, if one can find a Lax pair of any system, the system is integrable.

In general, one deals with an equation of the form

$$u_t = F(u) \quad \text{with} \quad u = u(x,t) \quad . \tag{1.1}$$

The general function F has no time dependence but may depend on the field u and its space derivatives. The Lax pair approach consists of finding suitable operators L and M such that the equation of motion may be written in the form

$$L_t - [L, M] = 0 \quad , \tag{1.2}$$

where the subscript characters denote differentiation. The linear operator L acts on the Hilbert space of states spanned by ϕ and obeys the eigenvalue equation

$$L\phi \;\; = \;\; \lambda\phi \quad , \tag{1.3}$$

where ϕ is an eigenfunction of L that may depend on the u field. We assume that the time evolution of the state ϕ satisfies the linear equation

$$\phi_t = M\phi \quad , \tag{1.4}$$

and take the eigenvalue λ to be real and time-independent [1]. The eigenvalue equation may now be differentiated with respect to t to produce

$$\begin{aligned} (L-\lambda)\phi_t + L_t\phi &= \lambda_t\phi \quad \Leftrightarrow \\ LM\phi - \lambda M\phi + L_t\phi &= 0 \qquad \Leftrightarrow \\ (LM - ML)\phi + L_t\phi &= 0 \end{aligned} \tag{1.5}$$

[1]This may be easily demonstrated if we take the operator L to be self-adjoint

By suitable choice of operators this equation becomes the equation of motion for the linear system. The difficulty in this method lies in the arbitrary choice of the L operator. In the original paper by Lax, the operator had already been identified in the Kruskal et al treatment of the Kortweg de-Vries model. After L has been found, it is a trivial task to find an M operator so as to reproduce the equation of motion in the form of (1.2). It has to be pointed out that the choice of M is not unique since it is defined up to a function that commutes with L.

The Lax pair equation implies that the time evolution doesn't change the spectrum of L thus making it possible to identify the eigenvalues of the operator L with the integrals of motion.

The Lax pair will be greatly used within this thesis. It provides an elegant demonstration of integrability and avoids cumbersome expressions. It will be used to express the complex sine-Gordon equation of motion in a compact form and to construct the infinite series of conserved charges.

Excellent reviews on the subject of integrability, the inverse scattering transform and the Lax pair formulation can be found in [9, 10, 11].

1.3 Solitons

Out of all stable solutions to non-linear evolution equations, soliton solutions are by far the most interesting. They describe well-defined, localised (solitary) waves exhibiting remarkable stability. This is a brief overview on a vast subject. More detailed treatments and introductory material can be found in the literature [12, 13, 14].

The first observation of a soliton is dated back to the beginning of the 19th century on a shallow water canal by John Scott Russell [15]. After a gap of almost 150 years, this wave which had in the meantime emerged as a solution of the integrable Kortweg de-Vries equation, appeared again in a study for the finite heat conductivity of solids. In a series of papers Kruskal and Zabusky [16, 17], studied the properties of this peculiar wave which they named soliton and showed that it was the Kortweg de-Vries equation again which was responsible for this solution.

Soliton solutions however are not a unique feature of the Kortweg de-Vries model. They appear in a huge range of different models all of which are integrable. This is a key point since most of their fascinating features are in fact associated with the infinite conserved quantities of integrable models.

The first remarkable feature of solitons is their stability. Although one would expect the wave to disperse with time, the soliton is created in such a way so that the wave stays focused and localised. This is the result of two opposing forces, a non-linear one which acts against the dispersion process which is in turn dictated by a linear force.

The second surprising feature of solitonic solutions is their scattering process. During scattering, the solitons, after a brief period of reconfiguration, emerge unaltered maintaining all physical characteristics such as speed, shape and volume. After the collision, no trace remains of the event but for a phase shift. No radiation or other mode is produced by the scattering process thus keeping the energy distribution localised.

The above properties make solitons ideal candidates for particle modeling. These localised energy lumps maintain their identity even after collision, pointing to the existence of conservation laws. In fact the stability of solitons can be attributed to the infinite number of conserved quantities associated with the symmetries of the integrable model. In addition, soliton stability may also be explained by topology as in the case of the sine-Gordon model where solitons solutions interpolate between different vacua thus demanding an infinite amount of energy to collapse.

Apart from mathematical modeling, solitons have also fascinating applications in technology. Their incredible stability makes them a favorite choice in communications as they minimise or even eliminate the need for signal boosting and do not suffer from pulse overlapping problems due to minimal broadening effects.

Solitons in the past few decades have become an integral part of field theory and led to the introduction of more complicated topological objects like vortices and instantons. Along with integrability, they have shown how non-linearity can give rise to well-behaved and completely-solvable systems that may be used to explain a variety of phenomena. One such model that has found plenty of applications and combines all the nice features of integrability whilst possessing soliton solutions is the sine-Gordon model.

1.4 An example of a completely integrable system: The sine-Gordon model

Second perhaps only to the Kortweg de-Vries equation, the sine-Gordon model is one of the most famous and well studied non-linear models in the literature. The theory is Lorentz-invariant and has found many an application throughout physics. The name comes from the similarity with the linear Klein-Gordon model, which arises as the linear approximation. It is a completely integrable field theory in (1+1) dimensions, describing a real scalar field with a sine-type interaction. The corresponding Lagrangian density is

$$\mathcal{L}_{SG} = \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi - \frac{1}{\beta^2}\left(1 - \cos(\beta\phi)\right) \ , \tag{1.6}$$

where ϕ is the scalar field and β a real coupling constant. The sine-Gordon equation emerges as the Euler-Lagrange equation of the above Lagrangian

$$\partial^{\mu}\partial_{\mu}\phi + \frac{1}{\beta}\sin(\beta\phi) = 0 \ , \tag{1.7}$$

and accepts soliton solutions of the form

$$\phi = 4\arctan\left(\exp\left(\pm\frac{1}{\beta}(\cosh(\theta)x - \sinh(\theta)t)\right)\right) \ , \tag{1.8}$$

where θ is the rapidity of the soliton.This indeed describes a localised wave traveling with velocity $V = \tanh(\theta)$ without any change in form (Fig. 1.1). The stability of the solution is due to the infinite number of conservation laws. The model is completely integrable and the sine-Gordon equation appears as the zero curvature condition

$$[L, M] = 0 \ , \tag{1.9}$$

of a connexion

$$L = \partial_1 + i\left(\frac{\beta}{4}\partial_0\phi\sigma_3 + \sinh(\theta)\cos(\frac{\beta\phi}{2})\sigma_1 + \cosh(\theta)\sin(\frac{\beta\phi}{2})\sigma_2\right) \ , \tag{1.10}$$

$$M = \partial_0 + i\left(\frac{\beta}{4}\partial_1\phi\sigma_3 + \cosh(\theta)\cos(\frac{\beta\phi}{2})\sigma_1 + \sinh(\theta)\sin(\frac{\beta\phi}{2})\sigma_2\right) \ , \tag{1.11}$$

where σ_i are the standard Pauli matrices. One of the most fascinating things of this model is that the solutions are also stable for topological reasons. The degenerate

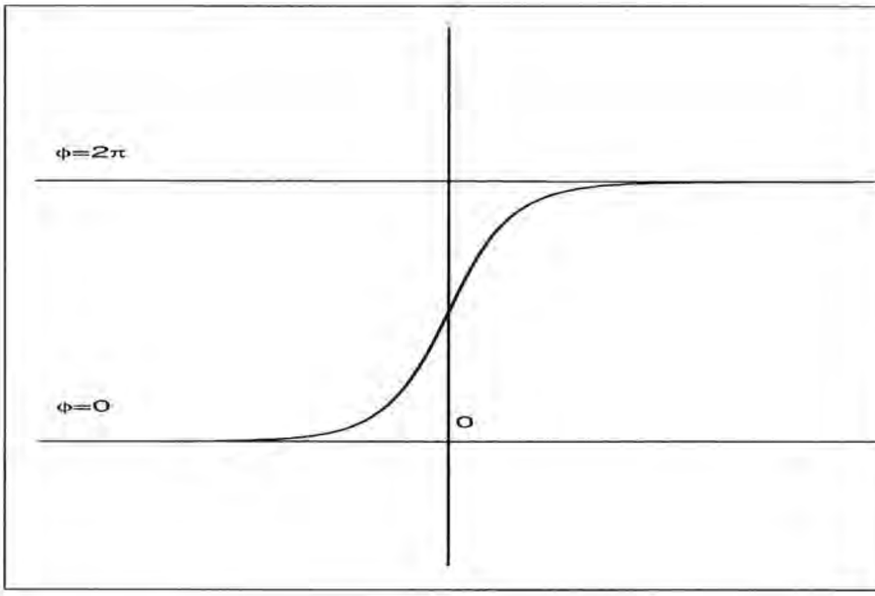

Figure 1.1: The sine-Gordon soliton

vacuum structure of the theory can be easily spotted. An infinite number of constant solutions satisfy the equation of motion minimizing the potential term

$$\phi = \frac{2\pi}{\beta} n \quad , \quad n \text{ integer} . \tag{1.12}$$

Through this degeneracy, topological aspects of the model arise. Solutions (kinks) that interpolate between different vacua are stable because an infinite amount of energy would be needed for their collapse. This is more elegantly expressed through the topological charge N, an integer, that corresponds to exactly this interpolation of the solution between vacuum values at infinity

$$Q_{top} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \partial_1 \phi = \frac{1}{2\pi} \left[\phi \,|_{-\infty} - \phi|_{+\infty} \right] = N \quad , \quad N \,\epsilon\, \mathbb{Z}. \tag{1.13}$$

The conservation law of the topological charge is not derived by a symmetry transformation, i.e. it is not a Nöether current. It exists independently and is associated with the vacuum structure. The topological charge as expressed for solitons also implies the existence of anti-solitons. For each soliton, there exists a topologically distinct solution that carries the opposite topological charge (antikink) having the exact same properties. An excellent study of the sine-Gordon equation and its properties can be found in [18].

Multi-soliton solutions also exist and have been studied extensively. Their scattering process demonstrates clearly the remarkable "transparency" feature where the solitons pass through each other without changing their original shape (Fig. 1.2 and Fig. 1.3).

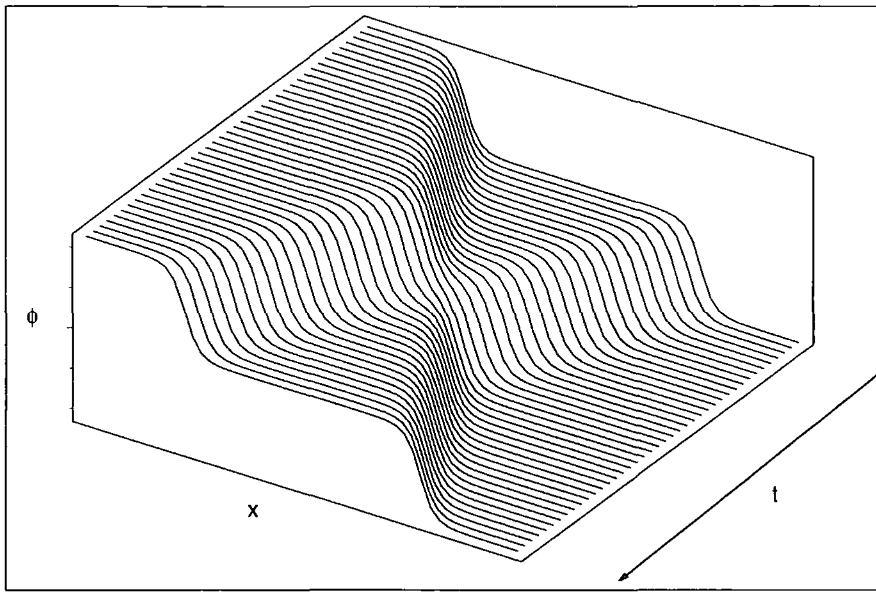

Figure 1.2: Kink-Kink collision

Multi-solitons solutions can be derived by the use of the Bäcklund transformation. It is a powerful method of obtaining more complicated solutions from already existing ones. The same technique will also be used in the following chapter to generate solutions for the complex sine-Gordon model and that is why a short description here is useful. We begin with the equation of motion of (1.7), which in lightcone coordinates is

$$\partial\bar{\partial}\phi + \sin(\phi) = 0 \ . \tag{1.14}$$

The coupling constant is set to $\beta = 1$ for simplicity. Consider the following set of equations

$$\partial(\phi_1 + \phi_2) = 2\delta \sin\left(\frac{\phi_2 - \phi_1}{2}\right) \tag{1.15}$$

$$\bar{\partial}(\phi_1 - \phi_2) = \frac{2}{\delta} \sin\left(\frac{\phi_2 + \phi_1}{2}\right) \ , \tag{1.16}$$

where ϕ_1 and ϕ_2 are both solutions of (1.14). It is easy to see that by differentiating with respect to $\bar{z}$ and z respectively and by using the equations again to discard the derivative terms, the sine-Gordon equation is recovered. This set of equations are the Bäcklund transformation for the sine-Gordon model. They imply that if one

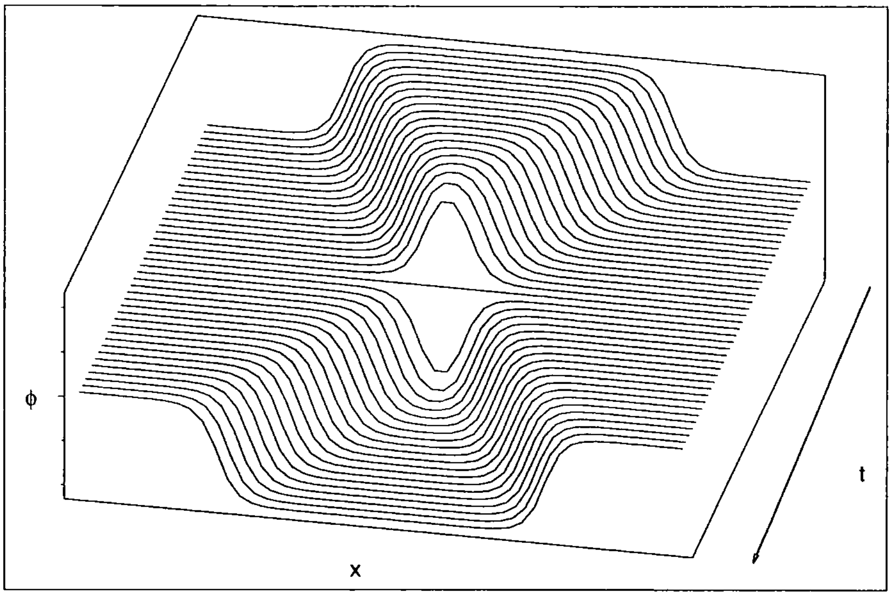

Figure 1.3: Kink-Antikink collision

solution is already known then by means of simple integration a second one may be obtained. The parameter δ is known as the Bäcklund parameter and appears in the new solution. To demonstrate this one may begin with the simplest solution to (1.14), namely $\phi_1 = 0$. Through the Bäcklund transformation, one can solve for ϕ_2 to discover the one-soliton solution of (1.8)

$$\phi_2 = 4\arctan\left(\exp(\delta z - \frac{1}{\delta}\bar{z} + C)\right) . \tag{1.17}$$

This method may be used recursively to construct higher solutions. However the integration becomes more difficult in the process. Instead it is better to use the "theory of permutability", a non-linear superposition technique is used to eliminate derivatives and simplify expressions (Fig 1.4)

One begins with a solution $S^{(0)}$ and uses the Bäcklund transformation twice with parameters δ_1 and δ_2 respectively to generate solutions $S_1^{(1)}$ and $S_2^{(1)}$. Now with starting point the latter two solutions the Bäcklund transformation is used again for each but this time the parameters are used in reverse order i.e. δ_2 and δ_1 respectively. The two solutions emerging coincide and may be written only in terms of $S_1^{(1)}$, $S_2^{(1)}$ and $S^{(0)}$, since derivative terms can be substituted by the original Bäcklund equations. If we begin with the solution $S^{(0)} = 0$ then in the first step we

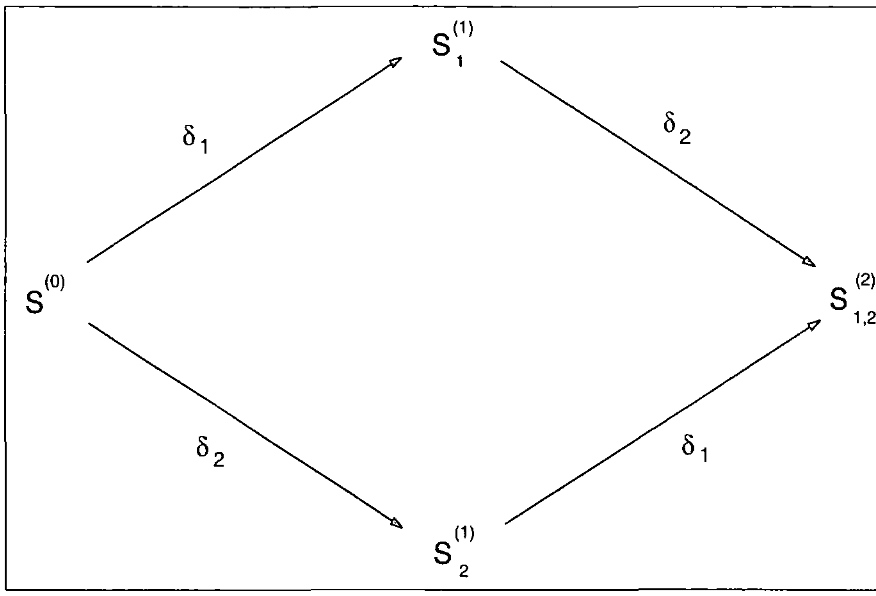

Figure 1.4: The non-linear superposition

produce two one-soliton solutions and in the next step a two-soliton solution

$$S^{(2)} = S^{(0)} \; 4 \; \arctan\left(\frac{\delta_1+\delta_2}{\delta_1-\delta_2}\tanh\left(\frac{S_1^{(1)}-S_2^{(1)}}{4}\right)\right) \; . \tag{1.18}$$

This is a general two-soliton solution with the arbitrary parameters δ_1 and δ_2 are related to the velocity of the two solitons. This formula can be generalised to n-th order providing a solid way of increasing the number of solitons in a solution.

There are two interesting choices for the parameters appearing in the two soliton solution of (1.18) that will be used to clarify some issues for the complex sine-Gordon model. The first one is

$$\delta_1 = -\frac{1}{\delta_2} = \sqrt{\frac{1-V}{1+V}} \; . \tag{1.19}$$

This choice corresponds to a soliton-soliton solution in the centre of mass frame. The collision process may be seen in (Fig. 1.2). The second choice is

$$\delta_1 = \frac{1}{\delta_2} = \sqrt{\frac{1-V}{1+V}} \; , \tag{1.20}$$

which corresponds to a soliton-antisoliton solution moving with equal and opposite velocities (Fig. 1.3).

The soliton-antisoliton solution is extremely interesting. It may be used to give rise to another set of solutions which correspond to bound states between a soliton and an antisoliton known as breathers (Fig. 1.5). These solutions are stable localised waves periodic in time that carry no topological charge. They appear from

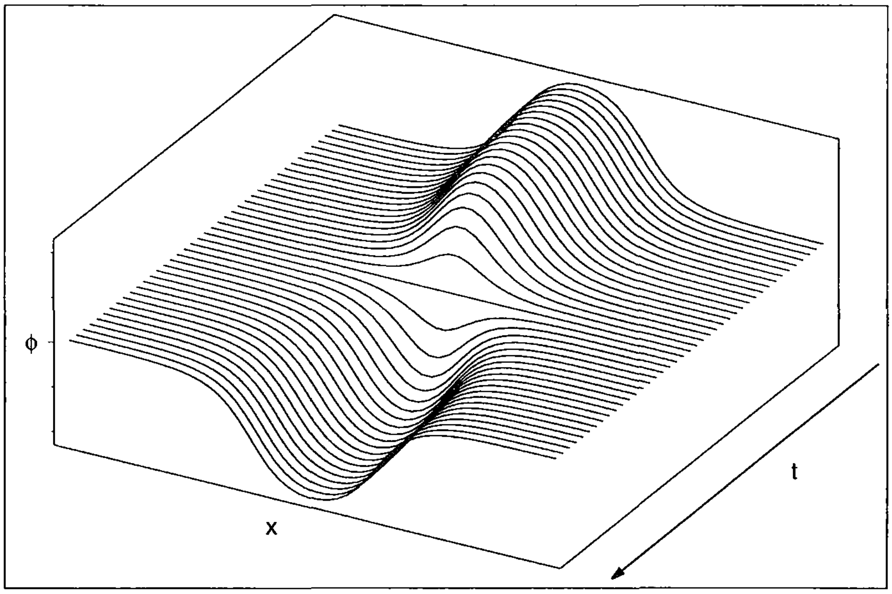

Figure 1.5: Breather solution

a soliton-antisoliton solution when the velocity parameter is analytically continued $V \to iV$. The absence of topology in such solutions has another effect. The breather corresponds to the particle of the theory. A small amplitude breather with specifically chosen parameters may be shown to correspond to small perturbation around the vacuum of the theory.

Soliton, antisoliton and breather solutions all belong to distinct topological classes. As we shall see in the next chapter this is not true for the complex sine-Gordon theory, where through transformations in the parameters we can switch between solutions.

1.5 The sine-Gordon model on a half line

In this section the sine-Gordon model is considered in the presence of a boundary. A boundary term is introduced to the sine-Gordon Lagrangian corresponding to an infinite energy barrier at $x = 0$. This modification raises the question under which circumstances does the model preserve integrability. Here we simply present the results. A more detailed review of the subject may be found in the papers of Saleur, Skorik and Warner [19] and of Ghoshal and Zamolodchikov [6]. Most of the results presented here come from the above mentioned papers. The full sine-Gordon Lagrangian containing the boundary term is

$$L_{SG} = \int_{-\infty}^{0} dx \left(\frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi - \frac{1}{\beta}\left(1 - \cos(\beta\phi)\right) \right) + \left[M_0 \cos\left(\frac{\beta}{2}\left(\phi - \phi_0\right) \right) \right]_{x=0} \tag{1.21}$$

The boundary term is evaluated at $x = 0$ and contains the real constants M_0 and ϕ_0. In their paper Ghoshal and Zamolodchikov showed that the model preserves integrability if one considers the boundary condition

$$\left[\partial_1\phi = M_0 \sin\left(\frac{\phi - \phi_0}{2} \right) \right]_{x=0} , \tag{1.22}$$

where the whole expression is evaluated at $x = 0$ and the field has been rescaled $\phi \to \frac{1}{\beta}\phi$. This is the most general condition consistent with integrability. It allows for the preservation of at least half of the conserved currents which are enough to render the model completely solvable. Inevitably the introduction of the boundary term destroys translation invariance and therefore momentum and higher-spin momentum-like quantities are not conserved. Neumann boundary conditions are obtained by setting $M_0 = 0$ which cause the sine-Gordon solitons to be reflected from the boundary as antisolitons. Another interesting limit is $M_0 = \infty$ which corresponds to Dirichlet boundary conditions making a soliton reflect as a soliton.

Using a technique similar to the method of images Saleur, Skorik and Warner calculated the phase delay induced by the scattering of solitons off the boundary. For this they used a three-soliton solution, fixing one of the solitons at the origin thus using it as a static background. The general formula for the phase delay that they derived is

$$a = \ln\left[-\epsilon \tanh^2\left(\frac{\theta}{2}\right) \tanh^2(\theta) \left(\frac{\tanh(\frac{1}{2}(\theta + i\eta)) \tanh(\frac{1}{2}(\theta - i\eta))}{\tanh(\frac{1}{2}(\theta + \zeta)) \tanh(\frac{1}{2}(\theta - \zeta))} \right)^{\pm 1} \right] . \tag{1.23}$$

The parameter $\epsilon = \pm 1$ is chosen in such a way so that the argument of the logarithm remains positive. The parameters ζ and η are directly related to the boundary parameters

$$M \cos\left(\frac{\phi_0}{2}\right) = 2\cosh(\zeta)\cos(\eta) , \tag{1.24}$$

$$M \sin\left(\frac{\phi_0}{2}\right) = 2\sinh(\zeta)\sin(\eta) , \tag{1.25}$$

and through the method of images to the Bäcklund transformation. Specifically, the Bäcklund parameter is related to ζ

$$\delta = e^{\zeta} \;, \tag{1.26}$$

whilst the parameter η appears as the limit of the known solution of (1.15) at the boundary

$$[\phi_1]_{x=0} = 2\eta \;. \tag{1.27}$$

Finally we present the boundary bound breather solutions for the sine-Gordon model. Just like the case in the bulk, boundary breathers are constructed through the analytic continuation of the velocity parameter, or in this case the rapidity $\theta \to i\theta$. Once again a static soliton background is needed. This three-soliton configuration has all the nice properties required for such a solution, i.e. finite energy levels and proper asymptotic behaviour. The solution of a boundary breather is

$$\begin{aligned} \phi_{br} = \;& 2\pi - \\ & -4\arctan\left[\frac{2\cot(\theta)\cot(\frac{\theta}{2})\sqrt{K}\cot(\frac{\phi_0}{4})e^{(1+\cos(\theta))x}\cos(t\sin(\theta))+e^{2\cos(\theta)x}K\cot^2(\frac{\theta}{2})+1}{2\cot(\theta)\cot(\frac{\theta}{2})\sqrt{K}e^{\cos(\theta)x}\cos(t\sin(\theta))+e^{x}\cot(\frac{\phi_0}{4})(e^{2\cos(\theta)x}K+\cot^2(\frac{\theta}{2}))}\right] \end{aligned} \tag{1.28}$$

with

$$K = \cot(\frac{\phi_0}{4} - \frac{\theta}{2})\cot(\frac{\phi_0}{4} + \frac{\theta}{2}) \;. \tag{1.29}$$

The above solution for the breather is real as long as K is positive. The spectrum of boundary bound breathers is continuous and only becomes discrete upon quantisation.

The boundary breather solution concludes the presentation of classical states for the sine-Gordon model. The results both in the bulk and on a half line will be useful when we come to analyse the complex sine-Gordon model in the following chapters. These will act as a guide in the chargeless limit where the complex sine-Gordon model collapses to the sine-Gordon equation. Topological aspects will not be subject to direct comparison since the topology of the chargeless complex sine-Gordon model is lost within a mapping. Nevertheless, the chargeless limit should reproduce all of the other familiar results from the sine-Gordon case.

1.6 Quantisation Methods

Considering classical theories which are completely solvable raises the question of how such theories behave when quantised. Quantum theories provide a much different picture than the corresponding classical models they originate from. Especially in the case of field theories the classical and quantum versions differ not only on the structure of physical states but in the treatment of the field variables themselves. Fields are no longer dynamic variables but operators acting on a Hilbert space of states and obeying commutation relations.

In the fourth chapter we shall examine the quantum case of the complex sine-Gordon model. Our objective will be to construct the quantum spectrum of boundary-bound states using the results of the theory in the bulk. In order to do that we shall use two methods which have different starting points but should in the end produce the same results. We end this introductory chapter with a brief presentation of these two methods.

1.6.1 The stationary-phase method

A number of different techniques have been developed in order to achieve the transition to the quantum level once the classical theory is well understood. The quantisation of such theories rarely leads to exact results and in general focuses on constructing quantum states out of the classical solutions with the addition of extra terms as "quantum" corrections. A number of such semi-classical methods exist. For example when quantising the anharmonic oscillator, one can use the Weak-Coupling method for time-independent solutions which is based on expanding the potential around the static classical solutions. The leading order is that of a harmonic oscillator, while the following anharmonic terms are small enough to be ignored. This method however is limited by the weak-coupling demand which is vital in order to ignore the anharmonic terms.

The method that will be used in the fourth chapter to obtain the energy spectrum of bound states is the stationary-phase approximation (SPA) . This is the natural extension of the WKB method of quantum mechanics for a field theory. It is based on the work of Dashen, Hasslacher and Neveu [3, 4] for the semi-classical quantisation of the sine-Gordon model. This method is not restricted by the weak-coupling demand and involves time-dependent periodic solutions. It is non-perturbative and may also be regarded as a generalisation of the Bohr-Sommerfeld quantisation rule. We choose to present briefly some key points of the SPA method as they will later be used in the half-line case. A pedagogical review on the subject may be found in the fifth chapter of [20] as well as in [21].

Consider the propagator associated with the transition of a system between two field configurations

$$\mathcal{G}(E) = \mathrm{Tr}\left(\frac{1}{E-H}\right) = \sum_n \frac{1}{E-E_n} \ . \tag{1.30}$$

The energy eigenvalues that appear as poles in $\mathcal{G}$ are exactly the energy levels of the bound states that we would like to find. Since this is a field theory an extra set of conditions have to be imposed at infinity

$$\phi(+\infty, t) = \phi(-\infty, t) = 0 \ . \tag{1.31}$$

The propagator $\mathcal{G}(E)$ can also be expressed as a functional integral

$$\begin{aligned} \mathcal{G}(E) &= \frac{i}{\hbar}\mathrm{Tr}\int_0^\infty dT \ \exp\left(\frac{i}{\hbar}(E-H)T\right) && (1.32)\\ &= \frac{i}{\hbar}\int_0^\infty dT \ \exp\left(\frac{i}{\hbar}ET\right) G(T) \ . && (1.33) \end{aligned}$$

where

$$G(T) = \mathrm{Tr}\left[e^{\left(-\frac{i}{\hbar}HT\right)}\right] = \int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar}S[\phi]\right) \ . \tag{1.34}$$

In general this is impossible to find exactly. The basic idea behind the SPA method is that the dominant contribution to the path integral in (1.34) comes from the periodic classical solution ϕ_{cl}. We shall call the period associated to such a solution T. Corrections to this approximation can be found by expanding the action functional $S[\phi]$ around the classical solution ϕ_{cl}

$$S[\phi] = S[\phi_{cl}] + \frac{1}{2}(\phi_a - \phi_{cl})\left[\frac{\partial^2 S[\phi]}{\partial\phi_a \partial\phi_b}\right]_{\phi=\phi_{cl}} (\phi_b - \phi_{cl}) + O((\phi_i - \phi_{cl})^3) \ , \tag{1.35}$$

assuming that fluctuations $(\phi_i - \phi_{cl})$ are small. The linear term in the expansion is zero as the solution ϕ_{cl} minimises the action functional whilst the quadratic term corresponds to quantum corrections. By substitution the propagator kernel now becomes

$$G(T) \simeq e^{\left(\frac{i}{\hbar}S[\phi_{cl}]\right)} \int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar}\frac{1}{2}\left[\frac{\partial^2 S[\phi]}{\partial\phi_a \partial\phi_b}\right]_{\phi=\phi_{cl}} \psi_a \psi_b\right) . \tag{1.36}$$

with $\psi_i = (\phi_i - \phi_{cl})$. The integral in the above expression comprises a generalisation of the familiar gaussian integral

$$\int_{\infty}^{+\infty} e^{-\frac{a}{2}x^2} dx = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} , \tag{1.37}$$

and can be solved to finally yield

$$G(T) \simeq e^{\left(\frac{i}{\hbar}S[\phi_{cl}]\right)} \det\left[\frac{1}{2}\left[\frac{\partial^2 S[\phi]}{\partial\phi_a \partial\phi_b}\right]_{\phi=\phi_{cl}}\right]^{-\frac{1}{2}} . \tag{1.38}$$

Solving for the determinant is not an easy task. It is an eigenvalue problem which is known as the stability equation

$$\left[\frac{1}{2}\frac{\partial^2 S[\phi]}{\partial\phi_a \partial\phi_b}\right]_{\phi=\phi_{cl}} \chi(x,t) = 0 . \tag{1.39}$$

The field $\chi(x,t)$ represents a small fluctuation around the classical field solution ϕ_{cl} and therefore $\phi_{cl} + \chi(x,t)$ satisfies the same equations of motion. The periodicity of the solution ϕ_{cl} dictates that

$$\chi_i(x, t+T) = e^{iv_i}\chi_i(x,t) , \tag{1.40}$$

where v_i is a phase factor known as the stability angle. There is an infinite number of χ_i solutions and therefore an infinite number of corresponding stability angles. They describe quantum fluctuations around the classical energy and appear as harmonic oscillator modes. Their contribution to the energy states can be seen if one examines the poles in $\mathcal{G}$. This involves some tedious calculations [4]. First the stability equation of (1.39) is solved. The form of G is then determined and substituted in (1.32). After some manipulation of the expression one finds the poles appearing when

$$\mathcal{F}(E) = \hbar\pi n \quad , \quad n = 1, 2, ... \tag{1.41}$$

with

$$\mathcal{F}(E) = S[\phi_{cl}] + S_{ct}[\phi_{cl}] + TE - \sum_0^\infty \frac{1}{2}\,\hbar\, v_i . \tag{1.42}$$

$$E = E_{cl} + E_{ct} + \sum_0^\infty \frac{1}{2}\,\hbar\,\frac{\partial v_i}{\partial T} \tag{1.43}$$

The sums appearing in the above expressions contain the stability angles and can be determined from solving the stability equation. It is because of these infinite sums which might be divergent that counter terms have to used. The terms S_{ct} and E_{ct} appear exactly for this purpose. For a renormalisable theory, such divergencies are expected to be cancelled by the introduction of such terms.

The quantisation condition (1.41) can be re-written in a more recognisable form. Let us define the quantity

$$\Delta = -S_{ct} + \sum_{i=0}^{\infty} \frac{1}{2} \hbar \, v_i \ . \tag{1.44}$$

Now $\mathcal{F}(E)$ may be written as

$$\mathcal{F}(E) = S[\phi_{cl}] - T\frac{\partial S[\phi_{cl}]}{\partial T} - \left(\Delta - T\frac{\partial \Delta}{\partial T}\right) \ , \tag{1.45}$$

where in this instance we have used $E_{cl} = -\frac{\partial S[\phi_{cl}]}{\partial T}$. By defining

$$S_{qu} = S_{cl} - \Delta \ , \tag{1.46}$$

we can finally write the quantisation condition in a more familiar form

$$S_{qu} - T\frac{\partial S_{qu}}{dT} = 2\pi n \ . \tag{1.47}$$

This is the generalised version of the Bohr-Sommerfeld quantisation condition

$$\mathcal{F}(E) = S + ET = \int pdq = 2\pi n \ . \tag{1.48}$$

From equation (1.45) it is easy to see that (1.47) in fact comprises of the quantisation rule for the classical action and a first order correction term described by

$$\Psi = \left(\Delta - T\frac{\partial \Delta}{\partial T}\right) \ . \tag{1.49}$$

We shall use this generalised quantisation condition when we come to calculate the boundary bound states spectrum on a later chapter. Although the Bohr-Sommerfeld rule for the classical action is quite straight forward, the calculation of the correction term is more subtle. The quantity Δ is not easy to calculate and shall prove to be divergent.

1.6.2 The Bootstrap approach

In a much different approach, exact results about the quantum states can be found through the bootstrap approach. The method is based on the work of A.B. Zamolodchikov and Al.B. Zamolodchikov [22] and involves the calculation of the exact form of the scattering matrix (S-matrix) through a set of constraints. Although in general the S-matrix is a complicated object, its form is quite restricted in the case of integrable theories in (1+1) dimensions. In such theories we can find the following useful properties

- **Non-production.**

  The number of incoming and outgoing particles remains the same. No new particles are produced through the scattering process.

- **Elasticity.**

  The initial and final sets of mass and momenta are the same. All scattering processes are completely elastic.

- **Factorisation.**

  The multi-particle scattering matrix can be written as a product of the two-particle S-matrices .

The last property implies that through this factorisation we only need to determine the form of the two-particle S-matrix . Let us define particle creation operators $A_a(\theta)$, where a denotes the type of particle and θ is the rapidity of the particle. The S-matrix can then be defined through

$$A_{a_1}(\theta_1)A_{a_2}(\theta_2) = S_{a_1,a_2}^{b_1,b_2}(\theta_1 - \theta_2)A_{b_2}(\theta_2)A_{b_1}(\theta_1) \ . \tag{1.50}$$

The S-matrix is subject to a set of constraints that will eventually determine its form. The first set of constraints originate from the charge conjugation, parity, and time-reversal symmetries

$$S_{a_1,a_2}^{b_1,b_2} = S_{\bar{a}_1,\bar{a}_2}^{\bar{b}_1,\bar{b}_2} = S_{a_2,a_1}^{b_2,b_1} = S_{b_2,b_1}^{a_2,a_1} \ . \tag{1.51}$$

Another set of constraints involves the use of the principles of analyticity, unitarity and crossing symmetry which can be expressed as follows

$$\begin{aligned}
&S_{ij}^{kl}(\theta) \in \mathbb{R} \ , \ \text{for } \theta \in \mathbb{I} && \text{Real analyticity} \\
&S_{ij}^{mn}(\theta) S_{mn}^{kl}(-\theta) = \delta_i^k \delta_j^l && \text{Unitarity condition} \\
&S_{ij}^{kl}(\theta) = S_{j,\bar{k}}^{l\bar{i}}(i\pi - \theta) && \text{Crossing condition}
\end{aligned} \tag{1.52}$$

Another condition imposing further restrictions on the form of the S-matrix is the Yang-Baxter equation. It follows from the ability to move around particle trajectories without changing the scattering amplitude of the process (Fig. 1.6). In terms

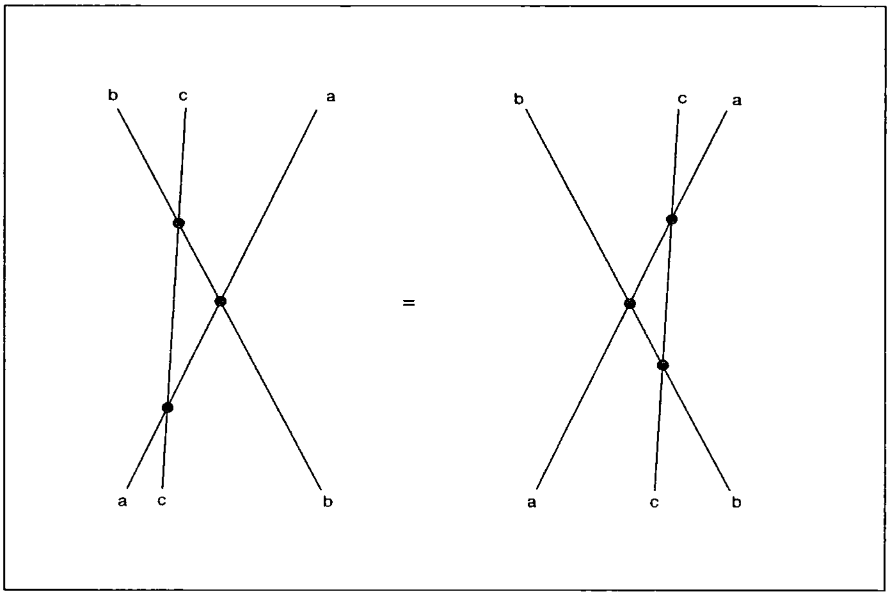

Figure 1.6: The Yang-Baxter equation

of S-matrices the Yang-Baxter equation is written as

$$S_{ac}^{ed}(\theta_{ac}) S_{eb}^{af}(\theta_{ab}) S_{df}^{cb}(\theta_{ab} - \theta_{ac}) = S_{cb}^{ed}(\theta_{ab} - \theta_{ac}) S_{ad}^{fb}(\theta_{ab}) S_{fe}^{ac}(\theta_{ac}) \ , \tag{1.53}$$

where

$$\theta_{ij} = \theta_i - \theta_j \ . \tag{1.54}$$

This is a powerful relation. In conjecture with symmetry aspects of the model at hand which give rise to particle multiplets, it can determine the two particle S-matrix . However if the particles of the theory are distinguishable by their mass and quantum numbers then the S-matrix is diagonal and therefore satisfies the Yang-Baxter equation identically.

One final constraint that is equally strong is the bootstrap equation. As mentioned above poles in the S-matrix may correspond to bound states. Although there are poles that can be interpreted differently (i.e. in terms of the Coleman-Thun mechanism [23]), we shall not investigate such cases. Bound states are formed when particles come together on a certain angle θ_{ab}^c known as the fusing angle (Fig. 1.7). Having analytically continued the S-matrix in terms of θ in order to consider

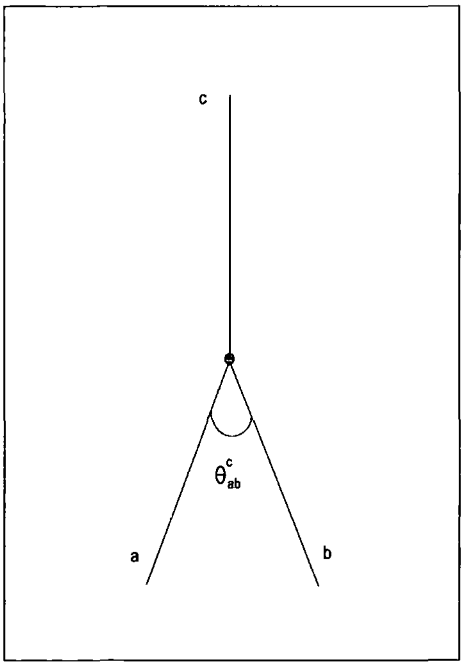

Figure 1.7: The fusing process

processes in the cross channel, the "physical strip", the region for all physical processes is introduced. The region is defined as $\Im(\theta) \in [0, \pi]$. Bound states occur only if the poles lie within the physical strip and more precisely on the $\Re(\theta) = 0$ axis. On this ground the contradiction with the non-production rule in the beginning of the section is avoided since θ is not real. The bound state appears now as a simple pole at $i\theta_{ab}^c$ for the forward and at $\pi - i\theta_{ab}^c$ for the crossed channel. The whole spectrum may be constructed once the basic particle is known. The bound state which is formed is again a different particle of the spectrum. It is on-shell and therefore can be made to fuse with the original particles to create new bound states. The process can be repeated until all states are accounted for.

As in the Yang-Baxter equation, we can again shift particle trajectories around a fusing process (Fig. 1.8). The result is the bootstrap equation

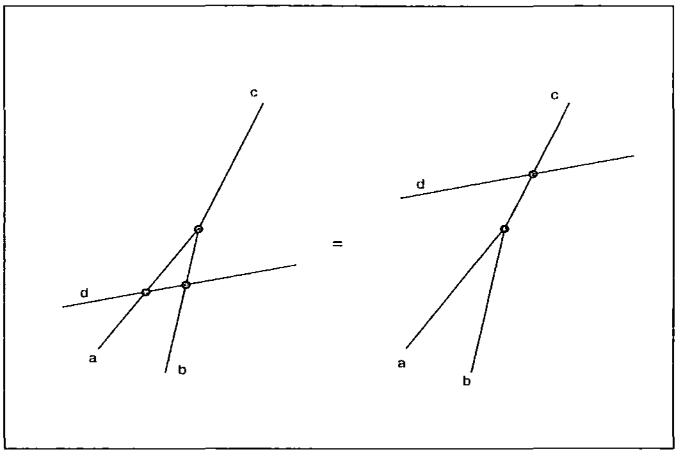

Figure 1.8: The bootstrap equation

$$S_{dc}(\theta_{dc}) = S_{da}(\theta_{dc} + \theta^{b}_{ca})S_{db}(\theta_{dc} + \theta^{a}_{cb}) \ , \qquad (1.55)$$

where θ^k_{ij} is the fusing angle of the process $ij \to k$, satisfying

$$\theta^k_{ij} + \theta^j_{ki} + \theta^i_{jk} = 2\pi \ . \qquad (1.56)$$

The bootstrap equation completes the set of constraints for the two-particle S-matrix. These are powerful enough to determine the exact form up to a total factor known as the CDD-ambiguity factor. More detailed reviews can be found in [24, 25]

A similar approach can be adopted when considering a theory on a half line. The introduction of a boundary also introduces the concept of the reflection matrix K in order to describe the particle-boundary interaction. The incoming and outgoing states are related through K (Fig. 1.9)

$$A_b(\theta) = K^a_b(\theta)A_a(\theta) \ . \qquad (1.57)$$

As in the case of the S-matrix , poles found in the reflection matrix may correspond to boundary-bound states [6, 26, 27]. Other explanations for the existence of poles are possible as in the bulk case, but a number of them should indicate the existence of new states.

In contrast however with the bulk case not all restrictions on the form of K apply. The presence of a boundary destroys translation invariance and therefore the parity condition of (1.51) does not hold. Moreover charge conjugation symmetry depends on the form of the boundary term and is not always true. Time reversal

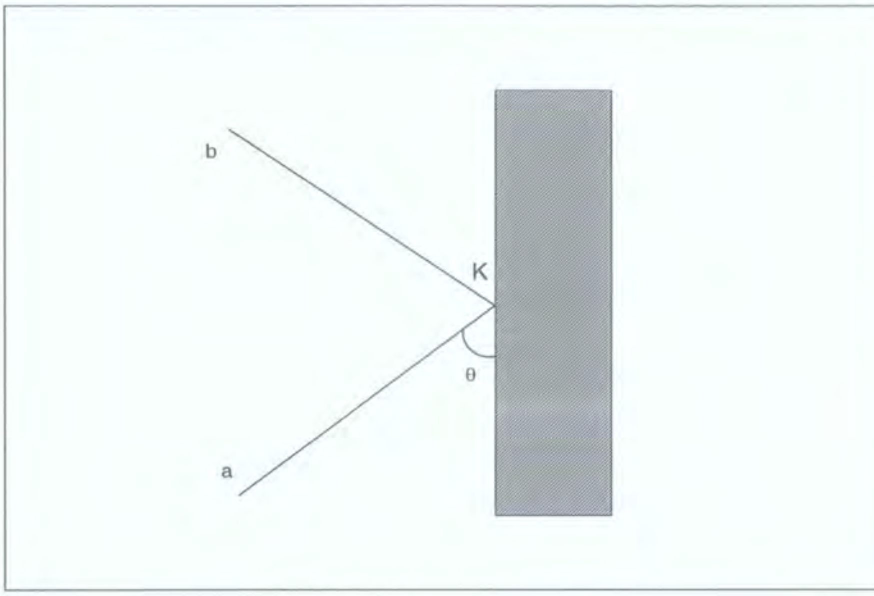

Figure 1.9: The reflection process

symmetry is however valid and presents us with the first restriction on K

$$K_a^b = K_b^a \ . \tag{1.58}$$

Unitarity and crossing symmetry conditions for the reflection matrix appear in the same way as for the S-matrix

$$\begin{array}{ll} K_a^c(\theta)K_c^b(-\theta) = \delta_a^b & \text{Unitarity condition} \\ K_a^b(\theta)K_{\bar{c}}^{\bar{d}}(\theta - i\pi) = S_{ab}^{\bar{c}\bar{d}}(2\theta) & \text{Crossing condition} \end{array} \tag{1.59}$$

In the above expressions K_a denotes the reflection matrix of particle a and $\bar{a}$ is the antiparticle of a. The remaining restrictions persist although in a slightly altered form. The boundary Yang-Baxter equation is again associated with shifting trajectories in the two-particle scattering diagram in the vicinity of the boundary. As with the bulk case, for distinguishable particles K is diagonal and the boundary Yang-Baxter equation (Fig. 1.10)

$$K_a(\theta_a)S_{ab}(\theta_a+\theta_b)K_b(\theta_b)S_{ab}(\theta_a-\theta_b) = S_{ab}(\theta_a-\theta_b)K_b(\theta_b)S_{ab}(\theta_a+\theta_b)K_a(\theta_a) \ .$$

is satisfied identically. Henceforth we shall take K to be diagonal since this is the case for the model at hand. Finally in the boundary case one has two bootstrap equations. Both are related with shifting trajectories around the fusion process close to the boundary. The first is the boundary reflection bootstrap describing the fusion of two particles before and after reflecting off the boundary wall (Fig. 1.11)

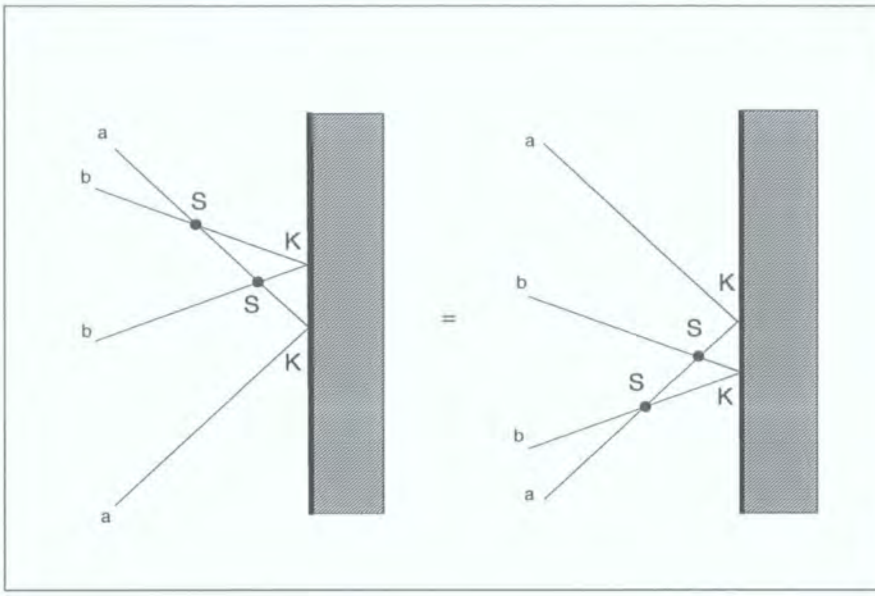

Figure 1.10: The boundary Yang-Baxter equation

$$K_c(\theta_c) = K_a(\theta_a)K_b(\theta_b)S_{ab}(\theta_a + \theta_b) \ , \tag{1.60}$$

with

$$\theta_a = \theta_c - i\bar{\theta}^b_{ac} \quad , \quad \theta_b = \theta_c - i\bar{\theta}^a_{bc} \ , \tag{1.61}$$

and $\bar{\theta} = \pi - \theta$. This is the basic relation that is used in order to build the whole spectrum of states. It can be used recursively to obtain the reflection factor of new particles and through their poles find new states. In order for this to work, the two-particle S-matrix of the bulk theory should be known. Crucially the reflection factor of the basic particle has to be obtained through some other means. This in fact is the most difficult step of the process, to propose a reflection matrix for the basic particle of the theory which is consistent with all the restrictions presented, has the correct classical limit and which contains a pole corresponding to a bound state in the spectrum.

The second bootstrap equation comes from the fact that the boundary may be in an excited state. Consider the creation of a boundary bound state (Fig. 1.12). A particle comes in at a specific angle $\theta_a^{\alpha\beta}$ which corresponds to a pole in K_a and fuses with the boundary thus changing its state $\alpha \to \beta$. This is reflected on the upper indices of the fusing angle. The boundary-bound bootstrap equation comes from shifting the trajectory of a particle reflecting from two differently excited boundary

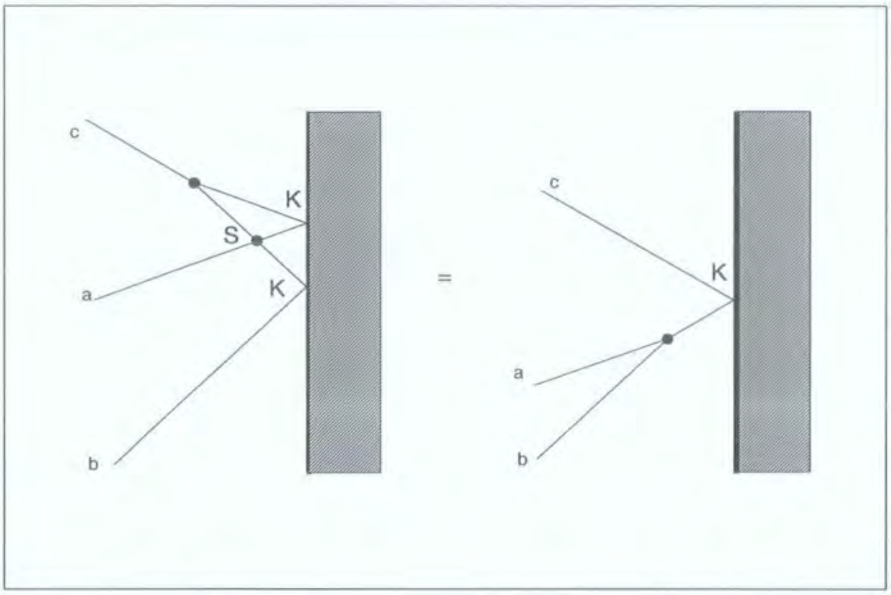

Figure 1.11: The reflection bootstrap equation

states (Fig. 1.13).

$$K_b^{(\beta)} = S_{ab}(\theta_b - \theta_a^{\alpha\beta}) K_b^{(\alpha)} S_{ab}(\theta_b + \theta_a^{\alpha\beta}) \ . \tag{1.62}$$

This concludes the set of restrictions that the reflection matrices should satisfy. More complex diagrams than those presented here may be written down but most can be interpreted in terms of the basic set.

Once again it has to be pointed out that not all poles in the reflection matrix correspond to bound states. For instance poles may be explained in terms of the Coleman-Thun mechanism for the boundary case (see for example [28]). Eliminating such poles and disregarding poles that do not belong in the physical strip, should lead to poles corresponding to bound states.

Introductory material and detailed discussions about the boundary bootstrap method may be found in a number of papers [29, 30, 31], along with applications to different systems [6, 32, 33, 34].

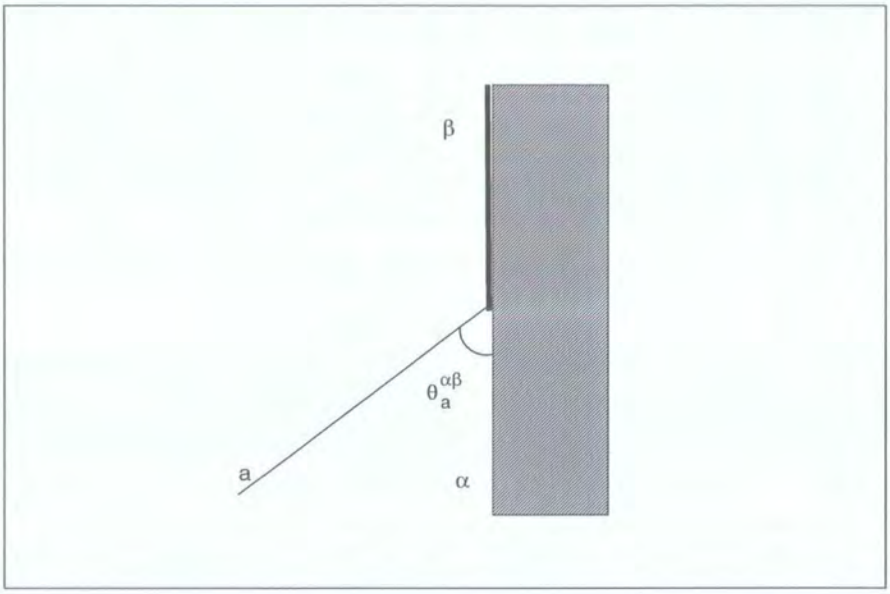

Figure 1.12: The boundary bound state

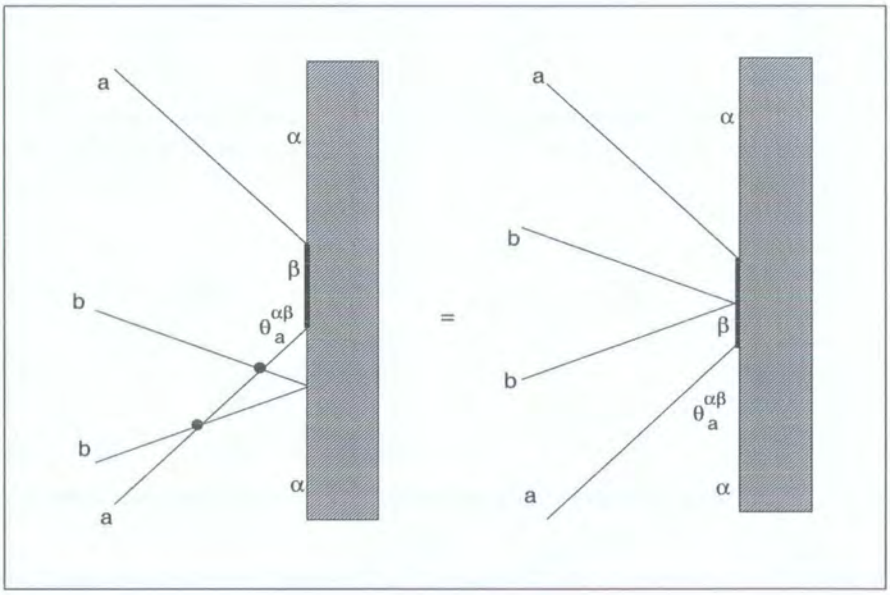

Figure 1.13: The boundary bootstrap

Chapter 2

The complex sine-Gordon model

2.1 Overview of the model

The complex sine-Gordon model[1] made its appearance in 1976 in two independent and unrelated papers. It was derived by Lund and Regge as a model describing relativistic string and vortices in a superfluid [35], and it was also obtained by Polmeyer in a reduction procedure of the $O(4) \simeq SU(2) \times SU(2)$ invariant chiral model which is identified with the one-dimensional σ-model [36].

The complex sine-Gordon equation admits soliton solutions that are both topological and non-topological. The non-topological solutions were written down by Lund and Regge [35], and by Getmanov in [37], each corresponding to a different sign of the coupling constant. These solutions carry a Nöether charge Q associated with the global $U(1)$ invariance of the evolution equation. In a later treatment by Shin and Park, the two sets of solutions were found to be related through a duality transform in the matrix potential framework [38].

The model's integrability was demonstrated by Lund using the inverse scattering method in [39]. On a later paper using Riemannian manifolds he also managed to express the non-linear complex sine-Gordon equation as an integrability condition [40].

In 1993 Bakas demonstrated that the model could be realised as a gauged WZW model, corresponding to the $SU(2)/U(1)$ coset model perturbed by the first thermal operator [41]. In this framework the theory at level n describes $\mathbf{Z}_n$ parafermions. The Krammers-Wannier duality between spin variables in the parafermion theory may be identified with the duality transform of the coupling constant and the theory may be expressed in an $SU(2)$ matrix form.

The complex sine-Gordon model is just the simplest case in a series of integrable generalisations of the sine-Gordon equation which belong to the group of non-abelian affine Toda equations. Out of all possible equations, only two series have a well defined action, i.e. have a positive kinetic term,a real potential and an S-matrix

[1]Getmanov has also studied a similar model which he referred to as complex sine-Gordon type II which includes an extra term in the Lagrangian. The model that is studied here is Getmanov's complex sine-Gordon type I.

description. The first are known as Homogeneous sine-Gordon theories (HSG) and are associated with a compact simple Lie group, while the second are known as the symmetric space sine-Gordon theories (SSSG) and are associated as their name implies with symmetric spaces. These theories have been studied extensively in [42, 43, 44, 45].

The complex sine-Gordon theory is quite fascinating from a mathematical point of view since it has a richer structure than the sine-Gordon equation which is recovered at the chargeless limit. It is also the only theory with a $U(1)$ gauge invariance which in the quantum limit has a completely factorisable S-matrix at tree level [46]. As the simplest case of a HSG theory, it may be used as a guide for more complicated generalisations. The introduction of the $U(1)$ charge distinguishes it from other extensively studied field theories, while its reduction to the sine-Gordon model in the chargeless limit places it close to them.

Additionally, the model has plenty of applications in different areas of physics apart from its original description of vortices in a superfluid and as a theory of parafermions. As was originally pointed out by Lund [40] the model provides useful insight on gravity on both classical and quantum level as it describes a massless field moving in the background geometry of a second field which has a sine-Gordon self-interaction. Indeed, the model in the massless limit was later shown to correspond to a stringy black hole through analytic continuation [47]. On a more applied level, the theory generalises the already successful theory by McCall and Hahn for optical pulses in a non-linear medium. McCall and Hahn successfully modeled the propagation of optical pulses in a non-linear medium by using the sine-Gordon model [48]. A few decades later Shin and Park [49] argued that a more suitable choice would be the complex sine-Gordon model in which more physical effects like inhomogeneous broadening and frequency detuning, which were previously ignored, would now be included within the framework of the field theory. Throughout this paper a close connexion with the theory of the optical pulses has been maintained. Key elements of the model are expressed in terms of physical quantities related to optical pulses. For completeness, we briefly present in an appendix the Maxwell-Bloch theory of optical pulses as well as demonstrate integrability of the model in the presence of a boundary with a more realistic choice of gauge. Through the introduction of a boundary term, an even more realistic modeling of optical pulses is achieved. The field theory description is not anymore restricted in an unbound non-linear medium, but now involves how pulses behave at the end of the wave guide.

2.2 Conventions

Before introducing the model, it is useful to present the notation that will be used throughout this thesis. This is a two dimensional model with one space x_1 (or simply x) and one time x_0 (or simply t) coordinate in Minkowski space with a the metric g = diag(1,-1) forming the basic measure

$$ds^2 = g_{\mu\nu}x^\mu x^\nu = x_0^2 - x_1^2 \; . \tag{2.1}$$

In some cases instead of the normal space-time coordinates x_0 and x_1, lightcone coordinates $z, \bar{z}$ will be used. The reason for this change is to present equations in the form in which they have originally appeared in related papers so that direct comparison may be possible. Lightcone coordinates will be related to the normal space-time ones through the following definition

$$z = \frac{1}{2}(t - x) \quad , \quad \bar{z} = \frac{1}{2}(t + x) \; . \tag{2.2}$$

In addition to simplify expressions and save space, derivatives are expressed in compact form

$$\partial_0^n = \frac{\partial^n}{\partial t^n} \quad , \quad \partial_1^n = \frac{\partial^n}{\partial x^n} \tag{2.3}$$

Analogous expressions are adopted for the derivatives in lightcone coordinates

$$\partial = \frac{\partial}{\partial z} \quad , \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} \; , \tag{2.4}$$

leading to the relations

$$\partial = \partial_0 - \partial_1 \quad , \quad \bar{\partial} = \partial_0 + \partial_1 \; , \tag{2.5}$$

that will be used extensively in the following chapters. Finally, we use the following conventions for the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; , \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; . \tag{2.6}$$

2.3 The complex sine-Gordon Lagrangian

The model has appeared in many different forms. In its original appearance the model was described by the Lagrangian

$$\mathcal{L} = \partial\phi\bar{\partial}\phi + 4\cot^2\phi\partial\eta\bar{\partial}\eta - 2\beta\cos 2\phi \ , \tag{2.7}$$

where ϕ and η are real fields and β a real coupling constant. The Lagrangian is singular at $\phi = n\pi$ which creates difficulties in the interpretation and full understanding of the model. In the following we will refer to this formulation as the trigonometric picture. The appearance of trigonometric functions, indicate a close connexion with the sine-Gordon theory. Indeed, when the field η is taken to be constant in (2.7), the sine-Gordon Lagrangian is recovered

$$\mathcal{L}_{SG} = \partial\phi\bar{\partial}\phi - 2\beta\cos 2\phi \ . \tag{2.8}$$

This form differs from the ordinary form of the sine-Gordon Lagrangian as it appears in the literature (1.6) through the double angle of the cosine as well as the undefined sign of the coupling constant β in the potential term. It is however a trivial difference involving the rescaling of both field and coupling constant. The Euler-Lagrange equations for this Lagrangian lead to the following equations of motion for the two fields

$$\begin{aligned}
\partial\bar{\partial}\phi + 4\frac{\cos\phi}{\sin^3\phi}\partial\eta\bar{\partial}\eta - 2\beta\sin 2\phi &= 0 \ , && (2.9)\\
\partial\bar{\partial}\eta - \frac{2}{\sin 2\phi}\left(\partial\phi\bar{\partial}\eta + \bar{\partial}\phi\partial\eta\right) &= 0 \ . && (2.10)
\end{aligned}$$

These are two real second-order coupled differential equations both maintaining the singular behaviour at $\phi = n\pi$.

The complex sine-Gordon model may also be expressed in a more compact form in terms of a single complex field. This constitutes an equivalent picture of the theory and will be referred to as the complex field picture. The complex sine-Gordon Lagrangian in this picture is

$$\mathcal{L}_{CSG} = \frac{1}{2}\frac{\partial u\bar{\partial}u^* + \bar{\partial}u\partial u^*}{1 - \xi^2 uu^*} - 4\beta uu^* \ , \tag{2.11}$$

with u now a complex field. The real parameter ξ, plays the role of the coupling constant, but is significant only on the quantum level. In the classical regime it corresponds to a general multiplicative scaling of the action as can be seen by rescalling the fields

$$u \to \frac{u}{\xi} \quad , \quad u^* \to \frac{u^*}{\xi} \; . \tag{2.12}$$

Since for the moment we are studying the model classically we shall set $\xi = 1$. The parameter will be restored later when we come to consider the quantum case. The remaining parameter β is a mass parameter that can be both positive and negative. This corresponds to dividing the theory into two distinct sectors, each with its own solutions, as will be explained in a later section. Through the variation of the action we obtain the equation of motion

$$\partial\bar{\partial}u + \frac{u^*\partial u\bar{\partial}u}{1-uu^*} + 4\beta u(1-uu^*) = 0 \; . \tag{2.13}$$

In this formulation the singular behaviour is now at $|u| = 1$. Solutions which start inside the disc $|u| = 1$ at early times cannot evolve into solutions outside the disc. Through the Lagrangian, one can easily calculate the classical energy of the solutions in the bulk

$$\mathcal{H}_{bulk} = \int dx \left(\frac{|\partial_0 u|^2 + |\partial_1 u|^2}{1-uu^*} + 4\beta uu^* \right) \; . \tag{2.14}$$

One of the key features that makes this model so interesting is that the complex sine-Gordon Lagrangian is invariant under $U(1)$ global transformations

$$u \to e^{i\alpha}u \quad , \quad u^* \to e^{-i\alpha}u^* \tag{2.15}$$

and α a real constant. This invariance gives rise to conserved $U(1)$ charges

$$Q = i\int dx \frac{u^*\partial_0 u - u\partial_0 u^*}{1-uu^*} \; , \tag{2.16}$$

that are carried by the solutions. The two Lagrangians of equations (2.7) and (2.11) are equivalent and are related through the transformation

$$u = \sin\phi e^{2i\eta} \quad , \quad u^* = \sin\phi e^{-2i\eta} \; . \tag{2.17}$$

The latter picture is the one that will be used throughout this thesis. The main reason for this choice is to avoid cumbersome expressions involving trigonometric functions. In the complex field formulation, there is a single equation of motion (2.13) in contrast with the set of coupled ones (2.9), making its analysis easier. The only drawback of this is perhaps the not so straightforward interpretation of the singular behaviour at $|u| = 1$, demanding extra caution when examining this specific limit.

2.4 The WZW interpretation

A different approach to the complex sine-Gordon model was offered by Bakas who demonstrated that the model may also be viewed as a gauged Wess-Zumino-Witten (WZW) model [41]. This is a more general picture for the complex sine-Gordon theory which encompasses the complex field picture. This is to be expected since the complex sine-Gordon model belongs to a series of generalisations of the sine-Gordon theory, the homogeneous sine-Gordon theories described by a gauged WZW action with an added potential term. The corresponding action principle is written as follows

$$S = S_{gWZW} + S_{pot} \ . \tag{2.18}$$

The action term S_{gWZW} is the well known gauged WZW action

$$\begin{aligned} S_{WZW} &= -\frac{1}{4\pi}\int_{\Sigma} dz d\bar{z}\ \mathrm{Tr}(g^{-1}\partial g g^{-1}\bar{\partial} g) \qquad (2.19) \\ &\quad -\frac{1}{12\pi}\int_{B}\ \mathrm{Tr}(\tilde{g}^{-1}d\tilde{g}\wedge\tilde{g}^{-1}d\tilde{g}\wedge\tilde{g}^{-1}d\tilde{g}) \\ &\quad +\frac{1}{2\pi}\int\ \mathrm{Tr}(-W\bar{\partial}gg^{-1}+\bar{W}g^{-1}\partial g+Wg\bar{W}g^{-1}-W\bar{W}) \ . \qquad (2.20) \end{aligned}$$

This action is defined in a three-dimensional manifold B whose boundary is our compactified normal two-dimensional space Σ. The field g is an $SU(2)$ group element and $\tilde{g}$ is the extension of g to the three dimensional manifold. The last term introduces gauge fields W and $\bar{W}$ which act as Lagrange multipliers. In addition, an extra deforming term S_{pot} is added

$$S_{pot} = \frac{\beta}{2\pi}\int\ \mathrm{Tr}(g\sigma_3 g^{-1}\sigma_3) \ . \tag{2.21}$$

The potential term breaks conformal invariance and thus gives rise to massive states. Varying the action yields the CSG equations of motion which can be expressed in a zero curvature form

$$[\partial + (g^{-1}\partial g + g^{-1}Wg + i\beta\lambda\sigma_3)\ ,\ \bar{\partial} + (\bar{W} - \frac{i}{\lambda}g^{-1}\sigma_3 g)\ ] = 0\ , \tag{2.22}$$

as shown by Park [50]. From the variation of the gauge fields W and $\bar{W}$, two constraint equations arise

$$\bar{\partial} g g^{-1} - g \bar{W} g^{-1} + \bar{W} = 0 \ ,$$
$$g^{-1} \partial g + g^{-1} W g - W = 0 \ , \tag{2.23}$$

which are important for making the identification with the CSG theory. The connexion between the $SU(2)$ matrix g and the complex field u of (2.13) is given by

$$g = \begin{pmatrix} u & -iv^* \\ -iv & u^* \end{pmatrix} \ , \tag{2.24}$$

where

$$v = -\sqrt{1 - uu^*} e^{-i\vartheta} \ , \tag{2.25}$$

is a field dual to u which is also a solution to complex sine-Gordon equation but with an opposite sign of β. The field variable ϑ should not be considered as an independent field but rather as an auxiliary field that is properly defined up to a constant through the constraint equations (2.23). Since g is an $SU(2)$ matrix, it follows that the relation

$$uu^* + vv^* = 1 \ , \tag{2.26}$$

is always true. We choose to examine the model in the gauge

$$W = \bar{W} = 0 \ , \tag{2.27}$$

which simplifies the constraint equations to the form

$$\partial\vartheta = -i \frac{u^* \partial u - u \partial u^*}{2(1 - uu^*)} \ , \quad \bar{\partial}\vartheta = -i \frac{u \bar{\partial} u^* - u^* \bar{\partial} u}{2(1 - uu^*)} \ , \tag{2.28}$$

whilst the equation of motion now becomes

$$[\partial - A, \bar{\partial} - \bar{A}] = 0 \ , \tag{2.29}$$

with

$$A = -(g^{-1}\partial g + i\beta\lambda\sigma_3) \ , \quad \bar{A} = \frac{i}{\lambda} g^{-1} \sigma_3 g \ . \tag{2.30}$$

This compact zero-curvature form of the equations of motion demonstrates the integrability of the model and will prove useful when we come to consider Bäcklund transformations and conserved quantities in later chapters. Not only does this formulation allow for generalisations to other related models (for instance by considering Lie algebras other than $SU(2)$), but also because it incorporates the fields u,v into the theory deals with different signs of the parameter β simultaneously. Different signs lead to different vacuum and soliton solutions indicating that the theory splits into two distinct sectors that must be treated independently. Let us consider the vacuum of the theory in order to address the problem. From the expression for the energy in the bulk (2.14) it follows that the most suitable candidate for a vacuum, would be a constant value for the field u that would force the kinetic term involving derivatives to vanish and at the same time minimize the potential term. It is clear to see from (2.7) and (2.17) that for $\beta > 0$ the vacuum is $u = 0$, while for $\beta < 0$ the vacuum should have $|u| = 1$ demonstrating that the $U(1)$ symmetry is spontaneously broken for such values of β. Soliton solutions also look very different depending on the sign of β as we shall see in the next section. In the WZW formalism however, both sectors are treated simultaneously as the diagonal and off-diagonal parts of the field variable g. In this context the fields u and v are both solutions to the CSG equation with opposite signs of β, and each corresponding to a different vacuum . Moreover, the two sectors are connected by a duality transform which interchanges the sign of the coupling constant β [38] and simultaneously interchanges the role of u and v. Thus the theory is invariant under the change

$$g \to g' = i\sigma_1 g = \begin{pmatrix} v & iu^* \\ iu & v^* \end{pmatrix} \quad , \quad \beta \to -\beta, \tag{2.31}$$

the latter representing a transform akin to the Krammers-Wannier duality of the Z_n parafermion theory [51]. Taking into account the invariance of the theory under this duality transform, we shall only consider the $\beta > 0$ sector which corresponds to the diagonal part of the matrix formalism. Returning to the vacuum, a suitable choice would be

$$g_{vac} = \begin{pmatrix} 0 & ie^{-i\Omega} \\ ie^{i\Omega} & 0 \end{pmatrix} . \tag{2.32}$$

This selection is consistent with the choices mentioned above. The diagonal $\beta > 0$ sector corresponds to the $u = 0$ vacuum , while the off-diagonal $\beta < 0$ to $|v| = 1$.

It may be noted that the apparent singular behaviour of the Lagrangian at $|u| = 1$ does not appear as a problem embedded in the theory but is a direct consequence of the fact that the gauge fields $W, \bar{W}$ are ill defined at the specific point.

2.5 Soliton solutions

As an integrable field theory, the complex sine-Gordon model admits soliton solutions. Because u and v appear in the definition of g (2.24) it is natural to find solutions for both simultaneously. As explained above, solutions for u will correspond to solitons of the CSG equation with $\beta > 0$, whilst solutions for v will correspond to solitons of the CSG with $\beta < 0$. Different techniques have been used to construct soliton solutions like the inverse scattering method [46] and the Hirota method [52]. However both methods yield results that are both cumbersome and difficult to manipulate. The Bäcklund transformation for the CSG model provides a more elegant way to obtain soliton solutions. The Bäcklund transformation is used to generate new solutions from ones we know already. The basic idea is to express the second order differential equations of motion as a set of first order differential equations which are easier to solve and which contain a known solution. The Bäcklund transformation for the complex sine-Gordon model can be written in terms of two $SU(2)$ matrix variables g and f [38] and the Bäcklund parameter δ

$$g^{-1}\partial g - f^{-1}\partial f - \frac{\delta\beta}{\sqrt{|\beta|}}[g^{-1}\sigma f, \sigma] \;=\; 0\,, \tag{2.33}$$

$$\bar{\partial} g g^{-1}\sigma - \sigma\bar{\partial} f f^{-1} + \frac{\sqrt{|\beta|}}{\delta}(g f^{-1}\sigma - \sigma g f^{-1}) \;=\; 0\,. \tag{2.34}$$

It is easy to show that both f and g satisfy the CSG equation as well as the constraint equation in the specific gauge choice. Taking f to be a known solution, one can generate a new solution through the equations presented above. One-soliton solutions can be derived by applying the Bäcklund transformation on the vacuum solutions g_{vac} of (2.32). Each sector of the theory provides us with two sets of two first order differential equations that can be integrated, in order to provide the one-soliton solutions. The off-diagonal elements that correspond to the $\beta < 0$ sector give

$$\begin{aligned} \partial_0 v - \sqrt{|\beta|}\, e^{i\Omega} \left(\delta - \frac{1}{\delta}\right)(1 - vv^*) &= 0 \\ \partial_1 v + \sqrt{|\beta|}\, e^{i\Omega} \left(\delta + \frac{1}{\delta}\right)(1 - vv^*) &= 0\,, \end{aligned} \tag{2.35}$$

that finally produce a different solution

$$v = -e^{i\Omega}\left(\cos(a)\tanh\left(2\sqrt{|\beta|}\cos(a)\frac{x - Vt}{\sqrt{1 - V^2}}\right) + i\sin(a)\right), \tag{2.36}$$

with Ω a real parameter associated with the vacuum of the theory, as in (2.32). This is the solution that was derived by Lund and Regge [35] when considering the $\beta < 0$ case.

Respectively for the diagonal elements of g which correspond to the $\beta > 0$ sector the set of equations is

$$\begin{aligned} \partial_0 u - \sqrt{\beta}\left(\delta\, e^{i(\vartheta+\Omega)} - \frac{1}{\delta}\, e^{-i(\vartheta+\Omega)}\right) u\sqrt{1 - uu^*} &= 0\,, \\ \partial_1 u + \sqrt{\beta}\left(\delta\, e^{i(\vartheta+\Omega)} + \frac{1}{\delta}\, e^{-i(\vartheta+\Omega)}\right) u\sqrt{1 - uu^*} &= 0\,. \end{aligned} \tag{2.37}$$

The one-soliton solution that emerges is

$$u = \cos(a)\exp\left(2i\sqrt{\beta}\sin(a)\,\frac{t - Vx}{\sqrt{1 - V^2}}\right)\operatorname{sech}\left(2\sqrt{\beta}\cos(a)\frac{x - Vt}{\sqrt{1 - V^2}}\right), \tag{2.38}$$

where V and a are real parameters associated with the velocity and charge of the soliton respectively. This solution was originally derived by Getmanov [52] for the $\beta > 0$ case. In addition an expression for the phase ϑ which appears in its dual field v, is also obtained

$$\vartheta = -\Omega - \arctan\left(\tan(a)\coth\left(2\sqrt{\beta}\cos(a)\frac{x - Vt}{\sqrt{1 - V^2}}\right)\right). \tag{2.39}$$

Both sectors are nevertheless interconnected through the duality transform of (2.31) so we shall choose to examine the $\beta > 0$ case. Moreover, since the mass parameter β is now considered positive, we shall make the following identification

$$m = 2\sqrt{\beta}\,, \tag{2.40}$$

for reasons of simplicity. We return to examine further the one-soliton solution of (2.38). Without loss of generality we take the velocity parameter $V = 0$ to obtain a static single soliton solution

$$u_{static} = \frac{\cos(a)\exp(im\sin(a)t)}{\cosh(m\cos(a)x)} \ . \tag{2.41}$$

This solution represents a sech-wave which oscillates in the internal $U(1)$ space with angular velocity $\omega = m\sin a$. The solution may be characterised as static but only in a sense that the centre of mass does not translate in space. From equations (2.14) and (2.16) we can calculate the mass and charge of this static solution

$$M = 4m\cos a \quad , \quad Q = 4\left(\text{sign}[a]\frac{\pi}{2} - a\right) \ . \tag{2.42}$$

The parameter a is directly associated with the charge and as expected the mass M depends on the charge through it. In the theory a appears only in trigonometric forms, and therefore should be considered as an angle variable. This indicates that the formula for Q which has previously appeared in the literature is only true for a certain region, namely $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$. If one plots Q as a function of a the result is a periodic pattern (fig. 2.1). To avoid confusion we shall consider a to lie in the region $0 \leq a \leq \frac{\pi}{2}$, unless stated otherwise.

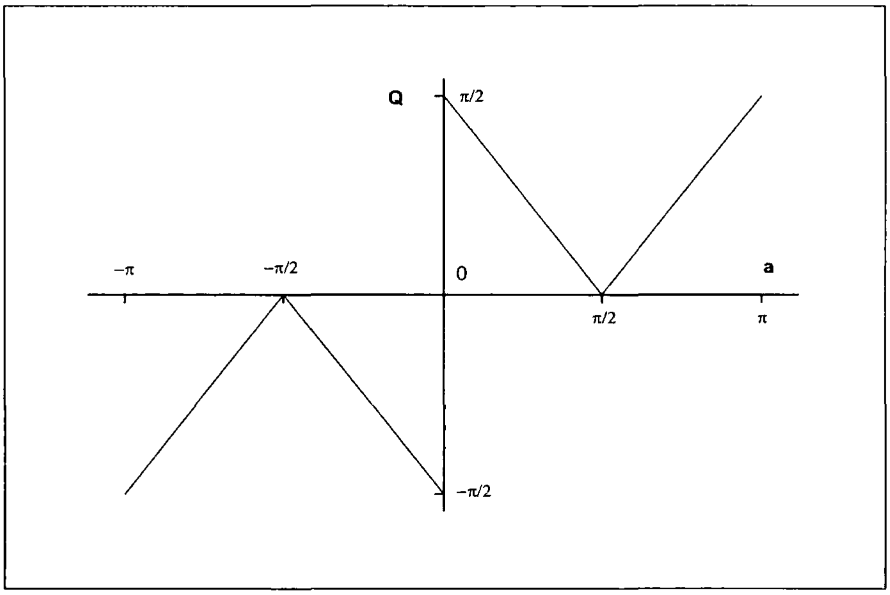

Figure 2.1: Soliton charge Q(a)

2.6 Multi-soliton solutions

A two-soliton solution can be obtained through a non-linear superposition technique based on the Bäcklund transformation ("theorem of permutability"). Starting from the vacuum of the theory and by the application of the Bäcklund transformation twice, a set of parameters $\{\delta_1, \delta_2\}$ is used respectively in each step. The same procedure is followed again where the two parameters are used in the opposite order. By demanding that the two results are equal, one ends up with an equation for the two-soliton solution in terms of single soliton and vacuum solutions (Fig. 1.4). For the complex sine-Gordon case this equation is in matrix form

$$g_{2s} = \sigma_3 \left(\delta_1 g_2 - \delta_2 g_1\right) g_{vac} \sigma_3 \left(\delta_1 g_1^{-1} - \delta_2 g_2^{-1}\right)^{-1} \ . \tag{2.43}$$

The matrix field variables g_k are of the general form of (2.24), with elements

$$u_k = \frac{N_k \cos(a_k) \exp(im \sin(a_k) \Theta_k)}{\cosh(m \cos(a_k) \Sigma_k)} \ , \tag{2.44}$$

$$v_k = -e^{i\Omega} \left(\cos(a_k) \tanh(m \cos(a_k) \Sigma_k) + i \sin(a_k)\right) \ . \tag{2.45}$$

The identification one must make is:

$$\Sigma_k = \frac{1}{2}\left(\delta_k + \frac{1}{\delta_k}\right) x + \frac{1}{2}\left(\delta_k - \frac{1}{\delta_k}\right) t \ , \tag{2.46}$$

$$\Theta_k = \frac{1}{2}\left(\delta_k + \frac{1}{\delta_k}\right) t + \frac{1}{2}\left(\delta_k - \frac{1}{\delta_k}\right) x \ , \tag{2.47}$$

where N_k is a total phase . As expected g_{2s} has the same general form of equation (2.24)

$$g_{2s} = \begin{pmatrix} u_{2s} & -iv_{2s}^* \\ -iv_{2s} & u_{2s}^* \end{pmatrix} \ . \tag{2.48}$$

The two-soliton solution and its complex conjugate are given by the diagonal elements of g_{2s}

$$u_{2s} = \frac{e^{i\Omega}\left(\delta_2 v_1^* - \delta_1 v_2^*\right)\left(\delta_1 u_1 - \delta_2 u_2\right) + e^{-i\Omega}\left(\delta_1 u_2 - \delta_2 u_1\right)\left(\delta_1 v_1 - \delta_2 v_2\right)}{\delta_1{}^2 - \left(u_1^* u_2 + u_2^* u_1 + v_1^* v_2 + v_2^* v_1\right)\delta_2 \delta_1 + \delta_2{}^2} \ , \tag{2.49}$$

$$u_{2s}^* = \frac{e^{-i\Omega}\left(\delta_2 v_1 - \delta_1 v_2\right)\left(\delta_1 u_1^* - \delta_2 u_2^*\right) + e^{i\Omega}\left(\delta_1 u_2^* - \delta_2 u_1^*\right)\left(\delta_1 v_1^* - \delta_2 v_2^*\right)}{\delta_1{}^2 - \left(u_1^* u_2 + u_2^* u_1 + v_1^* v_2 + v_2^* v_1\right)\delta_2 \delta_1 + \delta_2{}^2} \ , \tag{2.50}$$

while the off diagonal elements represent the dual field and its conjugate

$$v_{2s} = \frac{e^{i\Omega}\left(\delta_2 u_1^* - \delta_1 u_2^*\right)\left(\delta_1 u_1 - \delta_2 u_2\right) - e^{-i\Omega}\left(\delta_1 v_2 - \delta_2 v_1\right)\left(\delta_1 v_1 - \delta_2 v_2\right)}{{\delta_1}^2 - \left(u_1^* u_2 + u_2^* u_1 + v_1^* v_2 + v_2^* v_1\right)\delta_2\delta_1 + {\delta_2}^2}\,, \tag{2.51}$$

$$v_{2s}^* = \frac{e^{-i\Omega}\left(\delta_2 u_1 - \delta_1 u_2\right)\left(\delta_1 u_1^* - \delta_2 u_2^*\right) - e^{i\Omega}\left(\delta_1 v_2^* - \delta_2 v_1^*\right)\left(\delta_1 v_1^* - \delta_2 v_2^*\right)}{{\delta_1}^2 - \left(u_1^* u_2 + u_2^* u_1 + v_1^* v_2 + v_2^* v_1\right)\delta_2\delta_1 + {\delta_2}^2}\,. \tag{2.52}$$

The expressions above represent two-soliton solutions (Fig. 2.2) to the equation of motion and are related through the duality transformation of (2.31).

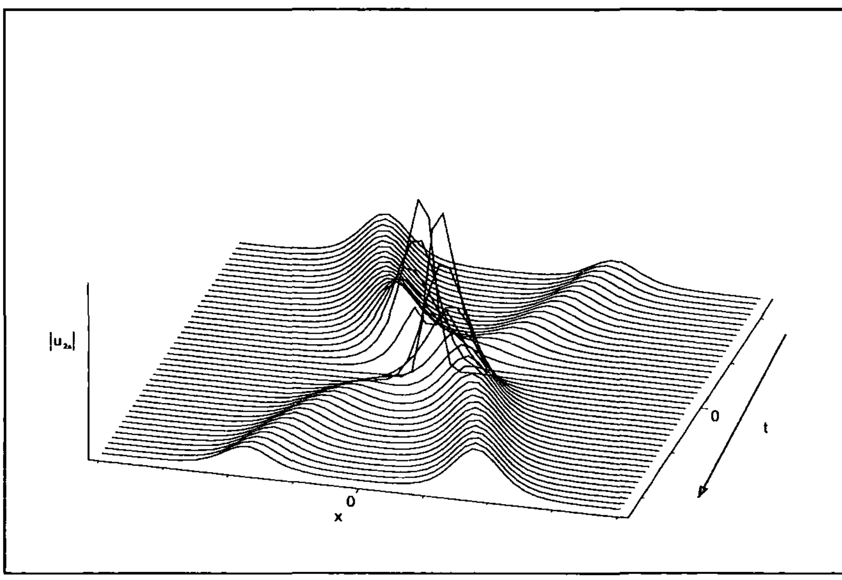

Figure 2.2: A two-soliton solution

Multi-soliton solutions can also be obtained by following the same technique. Instead of the vacuum solution, one can start from any given n-soliton solution S_n and add solitons through the method described above, ending up with a S_{n+2} solution. Nevertheless, expressions for multi-soliton solutions for this model are painfully large. Even the two soliton solution is so complicated (for disbelievers see Appendix B) that it is not possible to show that it satisfies the equations of motion directly. Although it is a trivial procedure to obtain a formula for multi-soliton solutions in terms of single soliton ones, its size and complexity makes it practically unusable apart from numerical calculations.

2.6.1 Soliton - Antisoliton duality

Thus far no mention of antisoliton solutions has been made. This is because antisolitons are not distinct classes of solution in the complex sine-Gordon model. Charged solitons are non-topological solutions, therefore a distinction between a soliton and an antisoliton is impossible. On the other hand chargeless solutions may be realised as topological solitons and identified with the sine-Gordon solitons.

The sine-Gordon theory appears as the limit of the CSG model when the charge parameter a is set to zero. We can substitute in the equation of motion of (2.13)

$$u = \sin\phi e^{2i\eta} \ , \tag{2.53}$$

where η is now a constant to recover the sine-Gordon model in the usual form

$$\partial_0^2\phi - \partial_1^2\phi + 2\beta\sin 2\phi = 0 \ . \tag{2.54}$$

The sine-Gordon theory has topological solitons (both kinks and antikinks) interpolating between its degenerate vacua. In contrast the CSG theory has a single vacuum for $\beta > 0$, and therefore its solitons are not topological in nature, but are stable because of integrability alone. The topological nature is hidden within the mapping of (2.53) and one has to be careful when trying to recover the sine-Gordon soliton as a limit of the CSG theory. Nonetheless a subtle remnant of the topology survives the mapping to the complex sine-Gordon theory. To see this consider how a SG soliton is mapped to CSG soliton. This is shown in Fig.2.3.

Consider now how the potential term behaves as x increases for the single soliton solution. As a function of u we can express the potential as

$$\sin 2\phi = 2\sin\phi\cos\phi = \pm 2u\sqrt{1-uu^*} \ , \tag{2.55}$$

where η has been ignored as a total phase. Note that we should take the branch cut with opposite signs on each side of the point $\phi = \frac{\pi}{2}$, $u = 1$. We shall see that the changing sign of the branch cut for a chargeless soliton will be important when we come to consider the theory with a boundary.

In some sense the topology of the chargeless u-soliton is embedded in the branch cut that appears at the singular point $u = 1$. The choice of branch corresponds to a different vacuum for ϕ and therefore to a different topological charge.

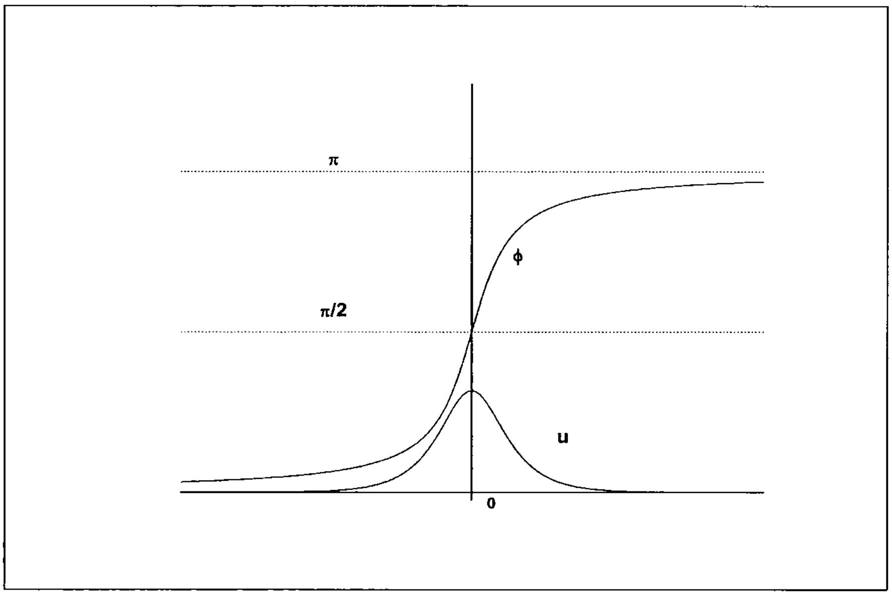

Figure 2.3: The ϕ and u solitons

However, when the charge parameter a is not zero, then the u-soliton does not reach the sick point $u = 1$ and remains non-topological. In this case no real distinction can be made between a soliton and an antisoliton. In the sine-Gordon case, the antikink solution is derived from the kink by changing the sign of the parameter δ of the Bäcklund transformation. This effectively corresponds to a parity and time reversal transformation which finally produces an antikink solution. Examining (2.44) we see that in the CSG case this change actually leads to the complex conjugate solution, by changing the sign of the complex phase. The change of sign in both t and x, can be cancelled by taking the charge parameter $a \to -a$. It is thus clear that instead of changing the sign of δ, one could effectively change the sign of a to derive an antisoliton. Since the soliton solution is a smooth function in a, the antisoliton is not a distinct object but can be identified with the soliton itself.

This also has an effect on the two-soliton solution. In previous treatments of the model, a soliton-soliton and a soliton-antisoliton solution were presented and treated as distinct solutions. Nevertheless this is not true. If we follow the same steps as in the sine-Gordon two-soliton solution then the solution u_{2s} of (2.49) corresponds to both a soliton-soliton and a soliton-antisoliton solution depending on the choice of sign for the Bäcklund parameter δ_2. The parameter δ_2 can be chosen in such a way as to describe one of the following

$$\delta_1 = -(\delta_2)^{-1} = \sqrt{\frac{1-V}{1+V}} \quad \text{soliton-soliton scattering} \tag{2.56}$$

$$\delta_1 = (\delta_2)^{-1} = \sqrt{\frac{1-V}{1+V}} \quad \text{soliton-antisoliton scattering} \tag{2.57}$$

Here we have taken the two solitons to have equal and opposite velocity (In general this is not the centre of mass since differently charged solitons have different masses, but it will be convenient for our discussion when we introduce a boundary later on). Also for reasons of simplicity we will refer to the soliton-soliton solution as u_{ss} ($\delta_2 = -1/\delta_1$) and to the soliton-antisoliton as u_{sa} ($\delta_2 = 1/\delta_1$).

However since no topological distinction exists between the soliton and antisoliton sector, it is possible to find a transformation of the parameters of the solution which effectively acts as a change of sign for the parameter δ_2. In fact, a set of transformations exists that maps u_{ss} to u_{sa} but we restrict ourselves to the simplest cases.

Before introducing the transformation, we need to introduce arbitrary shifts in x, which are crucial not only for this mapping but also later when we consider breathers and soliton reflections. The shifts appear in exponentials, so it is more helpful to consider the shifts in the following forms

$$\begin{aligned} K_i &= \exp\left(2\sqrt{\beta}\cos a_i \frac{x_i}{\sqrt{1-V^2}}\right) \\ J_i &= \exp\left(2i\sqrt{\beta}\sin a_i \frac{V y_i}{\sqrt{1-V^2}} + iR_i\right) ; \quad i = 1,2 \ . \end{aligned} \tag{2.58}$$

The parameters K_i, J_i are directly related with both V and a and correspond to the arbitrary initial positions in x, in the real (Σ_i) and imaginary phases (Θ_i) respectively, that appear in the one-soliton solution. Specifically the parameter K represents a translation in x, while the J parameter represents a phase shift in the internal $U(1)$ space. For reasons of simplicity, we include in the definition of J the total phase $N_k = \exp(iR_k)$ which appears in (2.44). Henceforth these parameters will be referred as phase shifts, since they are directly related to the time-delay effect of the scattering process.

Now that we have defined the arbitrary phase shifts we start with the soliton-soliton solution u_{ss} which comes from the two-soliton solution u_{2s} when we choose $\delta_2 = -1/\delta_1$. We consider the following transformation

$$a_2 \to -a_2 \ . \tag{2.59}$$

Although this is enough to change a single soliton to an antisoliton, this is not the case for the two-soliton solution. The phase shifts have also to be fixed in a specific way to complete the mapping between u_{ss} and u_{sa}

$$J_1 \rightarrow -J_1 \tag{2.60}$$

$$K_2 \rightarrow 1/K_2 \ . \tag{2.61}$$

This effectively changes the sign of δ_2 in the expression u_{ss} converting one of the solitons to an antisoliton. In contrast with the single soliton where the antisoliton can not be properly defined, in the two-soliton case there is a point of reference. A distinction between a soliton and an antisoliton can only be realised as a specific choice of the relative sign between the parameters a_1, a_2 and V which does not in any case lead to topologically distinct solutions.

The same mapping between u_{ss} and u_{sa} can also be achieved by making the following transformation

$$a_2 \rightarrow a_2 + \pi \ , \tag{2.62}$$

which effectively changes the sign of all trigonometric functions involving the parameter a_2 sending the solution $u_{ss} \rightarrow -u_{sa}$. This transformation will be used again on a later section when we come to consider soliton reflections, to demonstrate exactly the equivalence of the two sets of solutions.

Before continuing with the spectrum of the theory, a few things about the nature of the dual field v have to be mentioned. Unlike u which represents non-topological solutions but for the chargeless case, v is a topological object. The vacuum of the theory in the $b < 0$ case is $|v| = 1$ which implies that at infinity v does not necessarily go to the same vacuum. This may be seen directly by taking the limit of the expression (2.25) at $x = \pm\infty$. Since at infinity $u \rightarrow 0$, as expected v goes to a pure phase i.e. $\exp(i\vartheta)$. From the expression for ϑ (2.39), one can see that the limit is directly related to the charge parameter a. In the chargeless limit, any dependence on a vanishes and the field interpolates between vacua that differ by multiples of π. In this limit the theory is identified with the sine-Gordon theory and both the u and v solutions correspond to the sine-Gordon solitons.

2.6.2 Breather solutions.

There are conflicting views in the literature concerning the existence of breathers. In an early treatment of the model Getmanov presented breather solutions which were obtained through the usual method of analytically continuing in the velocity parameter V [52]. However Dorey and Hollowood dismissed the existence of such solutions and argued that breathers do not appear in the quantum spectrum of the model [53]. The problem arises because the transformation $V \to iV$ which is usually used to generate breathers from a two-soliton solution traveling with equal and opposite velocities, does not necessarily lead to a solution of the equations of motion. While the technique has been widely used before on other models, the fact that the CSG equation involves both u and u^*, implies that naively analytically continued solutions do not necessarily satisfy the equation of motion.

So it is not clear, for instance, that all the breather-like "solutions" of [38] do satisfy the CSG equations of motion. However, since the sine-Gordon is embedded in CSG model by taking u to be chargeless, the sine-Gordon breather solutions do satisfy the CSG equations of motion. In fact a family of charged, complex breather solutions does exist in CSG model. Although it is quite hard to actually check if a general breather solution satisfies the equation of motion, a trick can be used instead. We consider the two-soliton solution of (2.49) and we demand that this solution is even in V so that is effectively a function of V^2. Now the transformation $V \to iV$, doesn't change the reality properties of the solution but simply introduces an overall minus sign into the arbitrary parameter V^2, which is irrelevant. Making the solution even in V, means that a few restrictions have to be imposed. Firstly, the charge parameters have to be taken equal or opposite according whether $\delta_1\delta_2$ is plus or minus one respectively. Secondly, some of the arbitrary position parameters, have now to be fixed. However, up until now all the arbitrary phase shifts that appeared were either complex (J_i) or real (K_i) and there was no distinction between the shifts that originated from the space or time part of the phase. However, when constructing a breather solution, by analytical continuation of the V parameter a separation between the space and time shifts is induced. All shifts that associated with space end up as real parameters, while time shifts become imaginary. We can restrict ourselves to shifts only in the x direction. One could also consider more general phases which are complex and also depend on time and the parameter V. These however correspond to either $U(1)$ rotations or time translations which make their use obsolete. The arbitrary shift parameters are now both real

$$K_s = \exp\left(m\cos a_s \frac{x_s}{\sqrt{1+V^2}}\right)$$
$$J_s = \exp\left(m\sin a_s \frac{Vy_s}{\sqrt{1+V^2}}\right) \quad ; \quad s=1,2\ . \tag{2.63}$$

and should be compared with the general form of (2.58). In order to make a breather solution from the soliton-soliton case the following relations are required

$$K_1 = \pm\frac{1}{K_2} \quad \text{and} \quad J_1 = \mp\frac{1}{J_2}\ , \tag{2.64}$$

where the signs in these equation are correlated. It should be noted that more

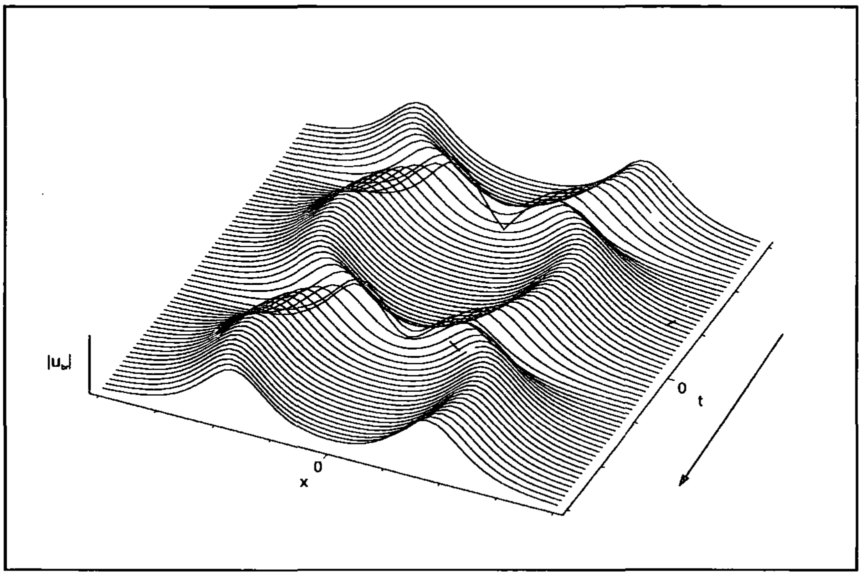

Figure 2.4: A breather solution

breather solutions may exist. It is possible that through certain restrictions a more general breather solution can be obtained, but a direct confirmation through the equations of motion is rather difficult.

2.6.3 Collapse of a Breather.

An analysis of the quantum CSG model [53] suggests that the soliton can be identified with the elementary particle since the vacuum of the theory and the one-soliton are not topologically distinct solutions. Evidence for this conjecture exists even in the classical picture. From our experience with the sine-Gordon model, we would expect to identify the particle with the lowest energy breather solution. It would seem to follow that the breather whose energy and charge correspond to that of a single particle should be equivalent to a single soliton. This remarkable fact can be shown as follows.

We consider the static single-soliton solution

$$u_{st} = \frac{\cos(a)\exp(im\sin(a)t)}{\cosh(m\cos(a)(x+x_0)}\ , \tag{2.65}$$

The mass and charge of the static soliton are given by (2.42)

$$M_s = 4m\cos(a_s) \quad , \quad Q_s = 4\left(\text{sign}[a_s]\frac{\pi}{2} - a_s\right)\ . \tag{2.66}$$

The breather is constructed from a two-soliton solution which has been analytically continued in V. Taking the solitons of the two-soliton solution far apart as $t \to \pm\infty$ we may consider them completely separated. Analytical continuation of $V \to iV$ yields a breather solution. The mass of such breather is twice the mass of a single soliton solution at velocity iV

$$M_B = \frac{8m\cos(a_B)}{\sqrt{1+{V_B}^2}}\ . \tag{2.67}$$

Following the same reasoning, the breather solution is effectively constructed from two one-soliton solutions, each with charge

$$Q_B = 4\left(\text{sign}[a_B]\frac{\pi}{2} - a_B\right)\ . \tag{2.68}$$

In order to have a chance of identifying the breather with the soliton, we demand that the mass of a breather is equal to the mass of a static single soliton and that their charges also coincide

$$M_s = M_B \quad , \quad Q_s = 2Q_B\ . \tag{2.69}$$

From the above relations, one can solve for the parameter V_B

$$V_B = \sqrt{\left(\frac{2\cos(a_B)}{\cos(a_s)}\right)^2 - 1} \; . \tag{2.70}$$

If this value is substituted into the breather, then the solution collapses to a static single-soliton carrying double the charge Q_B. In other words, the single-soliton can always be considered as a bound state of two single-solitons carrying half the charge. The argument can be used recursively so that a soliton can be regarded as an infinite collection of solitons carrying fractions of the original charge. At each level a soliton is identified with a breather emerging out of a soliton pair of half the original charge. In the classical picture this process can be carried out indefinitely, but in the quantum case the finite character of the mass states restricts this procedure.

This is not surprising since the static single-soliton of (2.65) can be viewed as a bound state due to the oscillation effect which creates a breather-like behaviour. This is consistent with the fact that any breather can collapse to this solution when the parameter V is properly fixed. It can therefore be realised as a breather solution after the collapse, exhibiting all of its former properties.

One point that has to be emphasized is that breathers constructed with the method described in the previous section are not chargeless. This is due to the fact that the choice of the charge parameters a_i is such that both solitons that are combined to create a breather have the same charge. This is confirmed by the above demonstration in which a breather collapses to a single soliton solution which carries double the charge of the breather's solitons. Neutral breathers do exist but only at the chargeless limit and can be identified with the breathers of the simple sine-Gordon theory.

2.7 Two soliton solution with unequal charges

In the previous section we saw that for a specific binding energy a breather collapses to a single soliton of twice the charge of the constituent solitons of the breather. In this section we argue that this is a special case and that in general a breather solution at any binding energy is identical to a static two-soliton solution of unequal charges. To demonstrate this we shall use the rapidity variable ϑ instead of V by changing

$$V = \tanh\theta \ . \tag{2.71}$$

and then analytical continuing in $\vartheta \to i\vartheta$. This is crucial as now the speed parameter ϑ is an angle variable much like the charge, thus allowing for both to combine and act in a similar way. This may be seen in the expression of the mass of the breather

$$M_B = 8m\cos a_B \ \cos\theta \tag{2.72}$$

which may be rewritten as

$$M_B = 4m\cos(a_B - \theta) + 4m\cos(a_B + \theta) \ , \tag{2.73}$$

if we assume that $a_B > \theta$ and that $a_B + \theta < \frac{\pi}{2}$ so that both terms remain positive. (The collapse of a breather to a single soliton in the previous section corresponds to the choice $\theta_B = \frac{\pi}{2} - a_B$.) Similar formulae may be written down for all values of a_B and θ. Compared with the mass of a static soliton of charge a (2.65) this relation seems to imply that the mass of a breather is equal to the sum of two well separated static solitons of charge $a_B - \theta$ and $a_B + \theta$ respectively. This assumption turns out to be true as can be verified by comparing the explicit formulae for breather solutions and static two-soliton solutions. One can show that the breather solution with characteristic parameters a_b and θ_b is the same as the static two-soliton solution with charges $a_b + \theta_b$ and $a_b - \theta_b$ as long as the phase parameter of each solution are fixed in a specific way. First the phase shifts of the breather are fixed, so this is a solution

$$K_2^{(br)} = K_1^{(br)} \quad , \quad J_2^{(br)} = J_1^{(br)} \ . \tag{2.74}$$

A similar choice is needed for the two-soliton phase parameters

$$K_2^{(2s)} = K_1^{(2s)} \quad , \quad J_2^{(2s)} = J_1^{(2s)} \ . \tag{2.75}$$

Finally the identification of the two solutions is achieved by choosing

$$K_1^{(2s)} = K_1^{(br)} \quad , \quad J_1^{(2s)} = J_1^{(br)} = 1 \ . \tag{2.76}$$

Once again the non-topological nature of solutions allows for this identification.

This raises once again the question whether breathers do exist in this model. The answer depends much on the point of view. Breathers are not a distinct class of solutions since they can be viewed as static two-soliton solutions with unequal charges. The quantum case seems to agree with this picture as all bound states related to poles in the S-matrix are accounted for without the introduction of breathers [53]. On the other hand the process of constructing breathers is not trivial. A two-soliton solution that is analytically continued in V and is shown to satisfy the equations of motion through some constraints in the parameters is a completely different approach in obtaining solutions. The absence of topology allows for solutions that in other models would appear in distinct classes (like the sine-Gordon model), to merge into a single group within which they may interchange their roles. Through the soliton-antisoliton duality no distinction between them may be established, while the basic particle is the soliton itself which is also identified with the breather. A distinction relies only on definitions which are not absolute and which are borrowed from similar theories.

2.8 Discussion

The CSG model, although it appears as the simplest case of the homogeneous generalisations of the sine-Gordon theory, has a rich mathematical structure and therefore possesses some fascinating features.

In the first part of the chapter we have examined the spectrum of the theory in the bulk and written down explicit two-soliton solutions within the framework of the matrix potential. We also demonstrated how to construct breather solutions in an elegant way avoiding the problems that arise by the analytical continuation of the parameter V.

An interesting aspect of the model is the soliton-antisoliton duality which appears since the model does not have a degenerate stable vacuum. The CSG soliton solutions are not topologically distinct and can therefore be interchanged by a continuous variation of the charge parameter a. Nevertheless, the topological nature of the sine-Gordon theory can be recovered in the chargeless limit as was demonstrated in section 2.6.1. A direct consequence of the non-topological nature of the

CSG soliton is that the breather solution can collapse to a single soliton when the parameter V is properly fixed.

Chapter 3

The complex sine-Gordon model on a half line

3.1 Reflections.

Integrable models, as a rule, are studied on the whole line. This is a simple scenario, since the natural constraint that the fields tend to the vacuum values at infinity is consistent with the conservation of charges. This however raises an interesting question; how should an integrable theory be modified so that when considered on a confined or a semi-confined region, it preserves its integrable character. The introduction of boundaries in field theories has become an object of extensive study in recent years and this has led to considerable progress in boundary quantum field theories. Not only does it explore new aspects of the model at hand but it also provides a more realistic approach to physical problems [54, 55].

The introduction of a boundary is realised by integrating the action over a restricted spatial geometry and the addition of an extra potential term to the Lagrangian of the model at the boundary. Of course we are allowed to add any potential term to the Lagrangian. Nevertheless our choices are limited considerably if we demand that the model remains completely integrable. The problem of finding suitable boundary terms that preserve integrability translates to a problem of finding suitable boundary conditions that lead to conserved charges. Not all conserved currents are expected to be preserved. This reflects the nature of the boundary: an infinite-energy barrier that breaks the symmetry in the bulk, e.g. translation invariance. Thus momentum and momentum-like currents will not be preserved. Since however the model is integrable, losing half of the initial infinite set of conservation laws, leaves the model completely solvable. The preserved currents are associated with time translation and are energy-like.

Once a set of suitable boundary conditions has been found, one can calculate the respective boundary terms by forcing the variation of the boundary term to vanish. The variation of a generic boundary term will involve the integrable boundary conditions, leaving a fairly simple form that has to cancel.

In this chapter, we derive boundary conditions consistent with integrability. By using the method of abelianisation of the Lax pair, we shall calculate the lowest spin-charges. Boundary conditions will arise by demanding the conservation of the charges in the presence of a boundary.

Following that, we shall calculate the corresponding boundary term and examine its energy contribution and effect on the theory's vacuum. Aspects of scattering in the model will also be explored, effectively calculating the reflection factor for particles and the time delay for solitons. Last but not least, the necessary conditions for the existence of boundary bound states will be established. The results presented here will be greatly used in the following chapter where the quantum case with a boundary will be presented.

3.1.1 Abelianisation of the Lax pair and conserved currents.

We shall consider a boundary condition to have preserved the integrability of the CSG model, if we can still construct an infinite number of commuting conserved charges. As mentioned above in contrast with the theory in the bulk, the introduction of a boundary destroys the translation invariance of the model but preserves the time translation invariance. It is thus expected that the momentum will not be conserved, whilst the energy will. This situation also holds for the higher-spin conserved quantities. All energy-like, parity-even quantities can be conserved, unlike their momentum-like, parity-odd partners. Nevertheless, since there is an infinite number of conservation laws, the main goal would be to concentrate on the conservation of the parity-even quantities.

The calculation of the lower spin charges was originally performed by Bakas [41] and later again by Fernandez-Pousa and Miramontes [43] in the general context of homogeneous models. The same calculation but in a different gauge has also appeared in the literature [49]. In this section we perform the same calculation in order to demonstrate how the boundary condition naturally arises by the demand that the lower spin charges are conserved at the boundary. We shall later use the same technique in a different gauge for the theory of optical pulses.

The presence of the spectral parameter λ in the Lax pair of (2.29) implies the existence of infinitely many conserved currents in the bulk that can be determined through the method used by Turok and Olive [56]. This is achieved by performing a gauge transformation U

$$\mathcal{A} = UAU^{-1} + \partial UU^{-1} \ , \tag{3.1}$$

in such a way that the commutator of the transformed gauge fields $\mathcal{A}$ and $\bar{\mathcal{A}}$ of the Lax pair is zero. The equation of motion becomes

$$\partial_0(\bar{\mathcal{A}} - \mathcal{A}) = \partial_1(\bar{\mathcal{A}} + \mathcal{A}) \ , \tag{3.2}$$

where the normal time and space derivatives are used. In the theory in the bulk we integrate over x. The conserved charges are expressed in the following form as a function of the spectral parameter λ

$$Q(\lambda) = \int_{\infty}^{+\infty} (\bar{\mathcal{A}} - \mathcal{A}) dx \ . \tag{3.3}$$

In order to verify that $Q(\lambda)$ is indeed a conserved quantity we shall consider the time derivative of this expression

$$\frac{d}{dt}Q(\lambda) = \int_{-\infty}^{\infty} \partial_0(\bar{\mathcal{A}} - \mathcal{A}) \ dx = \int_{-\infty}^{\infty} \partial_1(\bar{\mathcal{A}} + \mathcal{A}) \ dx = \left[(\bar{\mathcal{A}} + \mathcal{A})\right]_{-\infty}^{\infty} \ . \tag{3.4}$$

Since at infinity the fields are taken to vanish so that $\mathcal{A}$ and $\bar{\mathcal{A}}$ approach a fixed value, it follows that the right-hand side vanishes. As $\mathcal{A}$ and $\bar{\mathcal{A}}$ can be expanded as an infinite Laurent series in λ, the coefficients of each power of λ, provides us with an infinite number of conserved charges.

When a boundary is introduced the left-hand side involving the spatial derivative does not vanish since now the integration takes place over the semi-infinite interval. Instead one is left with an equation of the form

$$\int_{-\infty}^{0} \partial_0(\bar{A} - A) \ dx = \left[(\bar{A} + A)\right]_{x=0} \tag{3.5}$$

where the left-hand side is evaluated at the boundary. Instead of demanding that the right-hand side vanishes, we instead ask that it can be expressed as a total time derivative with the help of suitable conditions, thus leading to a conserved quantity.

We begin by finding explicit expressions for "low-spin" conserved charges of the CSG model in the bulk by solving for the abelianising gauge transformation U order by order in the spectral parameter.

Let U be a general real $SU(2)$ matrix, with $\det(U) = 1$. The diagonal elements of U can be taken equal due to residual gauge freedom which leave $\mathcal{A}$ and $\bar{\mathcal{A}}$ in an abelian form. Thus U takes the form

$$U = \frac{1}{\sqrt{1-\chi\bar{\chi}}}\begin{pmatrix} 1 & \chi \\ \bar{\chi} & 1 \end{pmatrix} , \tag{3.6}$$

where χ is a function of the fields and should not be associated with the space variable. We demand that U diagonalises both A and $\bar{A}$ at the same time. The transformed fields lie both in the σ_3 direction and the non-zero diagonal elements can be identified with the conserved currents. Taking A to be

$$A = \begin{pmatrix} i\Lambda & E \\ -E^* & -i\Lambda \end{pmatrix}, \tag{3.7}$$

with $\Lambda = \beta\lambda$ and $E = i(u^*\partial v^* - v^*\partial u^*)$, we demand that the non-diagonal part of A vanishes

$$\begin{aligned} 2i\Lambda\chi + \chi^2 E^* + E + \partial\chi = 0 \ , \\ 2i\Lambda\bar{\chi} + \bar{\chi}^2 E^* + E + \partial\bar{\chi} = 0 \ . \end{aligned} \tag{3.8}$$

The conserved quantities can also be written in terms of χ and $\bar{\chi}$

$$J = -i\Lambda\frac{1-\chi\bar{\chi}}{1-\chi\bar{\chi}} - \frac{\chi E^* + \bar{\chi}E}{1-\chi\bar{\chi}} + \frac{\chi\partial\bar{\chi} - \bar{\chi}\partial\chi}{2(1-\chi\bar{\chi})} \ . \tag{3.9}$$

The same matrix U, should also diagonalise $\bar{A}$, which is given by (2.30)

$$\bar{A} = \begin{pmatrix} D & P \\ P^* & -D \end{pmatrix} , \tag{3.10}$$

where $D = uu^* - vv^*$ and $P = -2iu^*v^*$. The choice of E, P and D is not accidental. They actually represent the electric field, the polarization and the population inversion field variables respectively, when this theory is used to describe the propagation of optical pulses in a non-linear medium [57]. When U acts on $\bar{A}$, we again demand the off diagonal parts to vanish. Examining the matrix explicitly yields

$$\begin{aligned} \frac{i}{\lambda}(-2D\chi + P - P^*\chi^2) + \bar{\partial}\chi &= 0 \ , \\ \frac{i}{\lambda}(2D\bar{\chi} + P^* - P\bar{\chi}^2) + \bar{\partial}\bar{\chi} &= 0 \ . \end{aligned} \tag{3.11}$$

It is easy to see that these equations are equivalent to equations (3.8).

The diagonal part yields the other component of the conserved current

$$\bar{J} = \frac{i}{2\lambda}\left(2\frac{1+\chi\bar{\chi}}{1-\chi\bar{\chi}}D + 2\frac{1P^*\chi - \bar{\chi}P}{1-\chi\bar{\chi}}\right) + \frac{(\chi\bar{\partial}\bar{\chi} - \bar{\chi}\bar{\partial}\chi)}{2(1-\chi\bar{\chi})}. \tag{3.12}$$

In order to solve the two sets of equations (3.8) or equivalently (3.11), we consider an expansion of χ and $\bar{\chi}$ in powers of Λ

$$\begin{aligned} \chi &= \frac{\chi_1}{\Lambda} + \frac{\chi_2}{\Lambda^2} + \frac{\chi_3}{\Lambda^3} + \dots \ , \\ \bar{\chi} &= \frac{\bar{\chi}_1}{\Lambda} + \frac{\bar{\chi}_2}{\Lambda^2} + \frac{\bar{\chi}_3}{\Lambda^3} + \dots \ . \end{aligned} \tag{3.13}$$

The coefficients χ_i and $\bar{\chi}_i$ can be determined by direct substitution into (3.8) and (3.11), and by demanding that the coefficients in all powers of Λ vanish. Up to order $\mathcal{O}(\Lambda^{-2})$ one finds

$$\begin{aligned} \chi &= \left(\tfrac{i}{2\Lambda}\right) E + \left(\tfrac{i}{2\Lambda}\right)^2 \partial E + \left(\tfrac{i}{2\Lambda}\right)^3 (E^2E^* + \partial^2 E) + \dots \ , \\ \bar{\chi} &= \left(\tfrac{i}{2\Lambda}\right) E^* + \left(\tfrac{i}{2\Lambda}\right)^2 \partial E^* + \left(\tfrac{i}{2\Lambda}\right)^3 (EE^{*2} + \partial^2 E^*) + \dots \ , \end{aligned} \tag{3.14}$$

Now that χ and $\bar{\chi}$ have been defined, we can also express the conserved quantities as a series in λ. Each order of λ, provides a conserved quantity and since the series of λ in χ and $\bar{\chi}$ does not terminate, we thus have an infinite number of conserved quantities as expected from the integrability of the CSG model. The two components of the conserved current up to $\mathcal{O}(\lambda^{-2})$ can be read off as coefficients in the following expansion of J and $\bar{J}$

$$J = -\lambda\beta - \frac{i}{2\beta}EE^*\left(\frac{1}{\lambda}\right) - \frac{1}{8\beta^2}(E\partial E^* - E^*\partial E)\left(\frac{1}{\lambda^2}\right) + \dots \ , \tag{3.15}$$

$$\bar{J} = iD\left(\frac{1}{\lambda}\right) + \frac{1}{4\beta}(E^*P - EP^*)\left(\frac{1}{\lambda^2}\right) + \dots \ , \tag{3.16}$$

and it can be checked that this current is conserved explicitly from the equation of motion.

In the above we have constructed conserved currents that lead to conserved charges in the bulk. However, as we have previously argued, conserved charges on the half line are expected to take the form of an integral over a *parity-even* conserved current. The conserved currents above are neither parity even or odd. To rectify this we note that our system of equations and constraints possess a Z_2 invariance involving parity transformations which can be used to construct a "reflected" set of conserved currents. The "reflected" set of conserved currents is easily obtained through the substitution $\partial \to \bar{\partial}$ in the expressions (3.15) including those derivatives involved in the definition of E. The new set of currents $\tilde{J}$, $\tilde{\bar{J}}$, can now be combined with the former set to produce pure parity odd and even currents.

In the presence of a boundary only parity even quantities are conserved. The desired form of the equations to emerge is

$$\partial_0 \ (parity \ \ even) = \partial_1 \ (parity \ \ odd). \tag{3.17}$$

By combining the two sets of currents one can separate the odd and even quantities for all powers of λ

$$\partial_0 \left[(\bar{J} + \tilde{\bar{J}}) - (J + \tilde{J})\right] = \partial_1 \left[(J - \tilde{J}) + (\bar{J} - \tilde{\bar{J}})\right] \ . \tag{3.18}$$

We examine the λ^{-1} term in the expansion which gives

$$\partial_0 \left(EE^* + \tilde{E}\tilde{E}^* + 2\beta(D + \tilde{D})\right) = \partial_1 \left(\tilde{E}\tilde{E}^* - EE^* + 2\beta(D - \tilde{D})\right), \tag{3.19}$$

where $\tilde{E} = E(\partial \to \bar{\partial})$, etc. After integration over the semi-infinite interval, the right hand side representing the parity odd part is

$$\partial_0 \mathcal{W}(u, u^*) = \left(\frac{2\partial_1 u^*}{1 - uu^*}\right) \partial_0 u + \left(\frac{2\partial_1 u}{1 - uu^*}\right) \partial_0 u^* \ . \tag{3.20}$$

This is a total derivative provided that

$$\frac{2\partial_1 u^*}{1 - uu^*} = \frac{\partial \mathcal{W}}{\partial u} \ , \quad \frac{2\partial_1 u}{1 - uu^*} = \frac{\partial \mathcal{W}}{\partial u^*} \ . \tag{3.21}$$

The conserved quantity at hand, in terms of u and u^*, is then

$$\mathcal{H} = \int_{-\infty}^{0} \left(2\frac{|\partial_0 u|^2 + |\partial_1 u|^2}{1 - |u|^2} + 4\beta(2|u|^2 - 1)\right) dx - [\mathcal{W}]_{x=0} \ , \tag{3.22}$$

Since this quantity actually represents the energy of the system, $\mathcal{W}$ can be identified with the energy contribution of the boundary term.

When constructing the odd and even quantities of the λ^{-2} term, one ends up with

$$\partial_0 \left(\frac{1}{2}(E^*\partial E - E\partial E^* + \tilde{E}\bar{\partial}\tilde{E}^* - \tilde{E}^*\bar{\partial}\tilde{E}) - \beta(E^*P - EP^* + \tilde{E}^*\tilde{P} - \tilde{E}\tilde{P}^*)\right) =$$
$$\partial_1 \left(\frac{1}{2}(-E^*\partial E + E\partial E^* + \tilde{E}\bar{\partial}\tilde{E}^* - \tilde{E}^*\bar{\partial}\tilde{E}) - \beta(E^*P - EP^* - \tilde{E}^*\tilde{P} + \tilde{E}\tilde{P}^*)\right) \tag{3.23}$$

Once more the parity-odd right hand side which after integration yields

$$-2\,\frac{(\partial_1 u^*)\,(\partial_0^2 u+\partial_1^2 u)}{1-uu^*}-4\,\frac{(\partial_0 u^*)\,\partial_1\partial_0 u}{1-uu^*}+2\,\frac{(\partial_1 u)\,(\partial_0^2 u^*+\partial_1^2 u^*)}{1-uu^*}+4\,\frac{(\partial_0 u)\,\partial_1\partial_0 u^*}{1-uu^*}$$
$$+2\,\frac{u^*\,(\partial_1 u)^2\,\partial_1 u^*}{(1-uu^*)^2}-2\,\frac{u\,(\partial_1 u^*)^2\,\partial_1 u}{(1-uu^*)^2}-4\,\frac{u\,(\partial_0 u^*)\,(\partial_1 u^*)\,\partial_0 u}{(1-uu^*)^2}+2\,\frac{u^*\,(\partial_0 u)^2\,\partial_1 u^*}{(1-uu^*)^2}$$
$$-2\,\frac{u\,(\partial_0 u^*)^2\,\partial_1 u}{(1-uu^*)^2}+4\,\frac{u^*\,(\partial_0 u)\,(\partial_1 u)\,\partial_0 u^*}{(1-uu^*)^2}+4\,(u\partial_1 u^*-u^*\,\partial_1 u)\,\beta\ ,$$

should be written as a total time derivative in order to force the currents to be conserved at the boundary. We can eliminate any second order spatial derivatives of the fields by using the equations of motion of (2.13)

$$4\frac{\partial_0 u\partial_0\partial_1 u^*}{1-uu^*}-4\frac{\partial_0 u^*\partial_0\partial_1 u}{1-uu^*}-4\frac{(\partial_1 u\partial_1 u^*+\partial_0 u\partial_0 u^*)(u\partial_1 u^*-u^*\partial_1 u)}{(1-uu^*)^2}$$
$$-4\beta(u\partial_1 u^*-u^*\partial_1 u)+4\frac{\partial_1 u\partial_0^2 u^*}{1-uu^*}-4\frac{\partial_1 u^*\partial_0^2 u}{1-uu^*}\ , \tag{3.24}$$

We take advantage of the fact that we are free to add total time derivatives on this expression, since this represents a conserved quantity. The expression simplifies significantly by adding the following term

$$\partial_0\left(4\frac{\partial_1 u^*\partial_0 u-\partial_1 u\partial_0 u^*}{1-uu^*}\right)\ , \tag{3.25}$$

which yields

$$-4\,\frac{(\partial_1\,u^*)\,\partial_0^2 u}{1-uu^*}+4\,\frac{(\partial_1\,u)\,\partial_0^2 u^*}{1-uu^*}+4\,\frac{(\partial_0 u)\,\partial_1\partial_0 u^*}{1-uu^*}-4\,\frac{(\partial_0 u^*)\,\partial_1\,\partial_0 u}{1-uu^*}$$
$$+4\,\frac{(-u\partial_1\,u^*+u^*\,\partial_1\,u)\,((\partial_1\,u^*)\,\partial_1\,u+(\partial_0 u)\,\partial_0 u^*)}{(1-uu^*)^2}$$
$$+4\,(-u\partial_1\,u^*+u^*\,\partial_1\,u)\,\beta\ . \tag{3.26}$$

This expression is evaluated at $x=0$, so in order to rewrite this as a total derivative we search for boundary conditions that are of the form

$$\partial_1 u=F(u,u^*)\quad,\quad\partial_1 u^*=G(u,u^*)\ , \tag{3.27}$$

where F and G are functions of the fields not involving derivatives. By direct substitution of the above into (3.26) we get

$$4\,\frac{\left(2\,\frac{\partial G}{\partial u}(1-uu^*)+Gu^*\right)(\partial_0 u)^2}{(1-uu^*)^2}-4\,\frac{(\partial_0 u^*)^2\left(2\,\frac{\partial F}{\partial u^*}(1-uu^*)+Fu\right)}{(1-uu^*)^2}$$
$$8\,\frac{(\partial_0 u^*)\left(-\frac{\partial F}{\partial u}+\frac{\partial G}{\partial u^*}\right)\partial_0 u}{1-uu^*}+4\,\frac{(-uG+u^*\,F)\,GF}{(1-uu^*)^2}$$
$$+4\,(-uG+u^*\,F)\,\beta\ . \tag{3.28}$$

The expression above does represent a total derivative when all terms are forced to vanish by selecting suitable functions F and G. The two separate differential equations that appear involving the undefined functions

$$\begin{aligned} 2(1-uu^*)\frac{\partial F}{\partial u^*}+uF &= 0 \ , \\ 2(1-uu^*)\frac{\partial G}{\partial u}+u^*G &= 0 \ , \end{aligned}$$

can easily be solved to yield

$$F(u,u^*) = S_1(u)\sqrt{1-uu^*} \quad , \quad G(u,u^*) = S_2(u^*)\sqrt{1-uu^*} \ . \tag{3.29}$$

In addition, the last two terms in (3.28) imply that

$$F = \frac{u}{u^*}\, G \ , \tag{3.30}$$

Using the above relation and solutions of (3.29) into the remaining terms of (3.28), we can determine the remaining undefined functions S_1 and S_2. The final form of the boundary conditions is

$$\begin{aligned} \partial_1 u &= -Cu\sqrt{1-uu^*} \ , \\ \partial_1 u^* &= -Cu^*\sqrt{1-uu^*} \ . \end{aligned} \tag{3.31}$$

The real boundary constant C is defined by the theory and is responsible for the way fields react to the boundary. Consistency of the two equations in (3.31) implies that C should be considered a real parameter. When one makes the transformation described in (2.17), the new boundary conditions for the fields ϕ and η are

$$\partial_1\phi = -C\sin(\phi) \quad , \quad \partial_1\eta = 0 \ , \tag{3.32}$$

which clearly shows that C has to be real.

It has to be pointed out that (3.31) is not the only set of boundary conditions that can be derived. Although we restrict ourselves only to cases where the space derivatives of the fields appear, a number of isolated "Dirichlet"-like conditions also exist. A more detailed treatment of such boundary conditions will appear in the revised version of [1]. If we take the field u to be real, the system is reduced to the sine-Gordon equation with a boundary condition $\partial_1\phi = -C\sin\phi$. This is the subset of integrable boundary conditions of the sine-Gordon model presenting the Z_2 symmetry $\phi \to -\phi$.

3.2 Soliton scattering and boundary bound states

Since the necessary conditions for the integrability of the model have been established, we study the scattering of particles and solitons off the boundary. We begin this section with the effects of introducing a boundary potential to the vacuum of the theory. We continue with the scattering of particles and solitons and derive the phase shifts induced by the process. Finally, we investigate the necessary conditions for the existence of boundary bound states.

3.2.1 Vacuum

When a boundary term is introduced, the vacuum of the theory that we discussed in section (2.2), does not necessarily remain unchanged. It is exactly this contribution that needs to be carefully examined before any statements are made about the minimum energy configuration. Although the vacuum solution of the theory in the bulk is a strong candidate, soliton solutions could also be considered in the attempt to both minimize the energy functional and satisfy the boundary conditions of (3.31).

We begin by first determining the energy contribution of the boundary term. The full Lagrangian of the model is now

$$\mathcal{L}_{tot} = \mathcal{L} + \mathcal{L}_{\mathcal{B}}. \tag{3.33}$$

The boundary term $\mathcal{L}_{\mathcal{B}}$, can be determined by the variation principle of the total action. The variation of the $\mathcal{L}$ term yields

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial u}\delta u + \frac{\partial\mathcal{L}}{\partial u^*}\delta u^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu u)}\delta(\partial_\mu u) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu u^*)}\delta(\partial_\mu u^*)\ . \tag{3.34}$$

When the Euler-Lagrange equations are used, two terms survive since the model is considered in the semi-infinite interval where the fields do not vanish at the boundary

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu u)}\delta u\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu u^*)}\delta u^*\right)\ . \tag{3.35}$$

From the variation of the boundary term one has

$$\delta\mathcal{L}_{\mathcal{B}} = \frac{\partial\mathcal{L}_{\mathcal{B}}}{\partial u}\delta u + \frac{\partial\mathcal{L}_{\mathcal{B}}}{\partial u^*}\delta u^*\ . \tag{3.36}$$

The variation of the action vanishes when the remaining terms evaluated at the boundary are forced to cancel. The two interrelated equations that emerge are

$$\frac{\partial \mathcal{L}}{\partial(\partial_1 u^*)} = \frac{-\partial_1 u}{1-uu^*} = -\frac{\partial \mathcal{L}_B}{\partial u^*} \ , \tag{3.37}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_1 u^*)} = \frac{-\partial_1 u^*}{1-uu^*} = -\frac{\partial \mathcal{L}_B}{\partial u^*} \ . \tag{3.38}$$

By substituting the boundary conditions of (3.31), these can easily be solved for the boundary term

$$\mathcal{L}_B = 2C\sqrt{1-uu^*} \ . \tag{3.39}$$

We now consider the total energy of the system , now comprising of two parts

$$\mathcal{H}_{tot} = \mathcal{H}_{bulk} + \mathcal{H}_B \ , \tag{3.40}$$

where the term $\mathcal{H}_{bulk}$, represents the energy in the bulk and the second term $\mathcal{H}_B$ represents the energy contribution from the boundary

$$\mathcal{H}_B = -2C\sqrt{1-uu^*} \ , \tag{3.41}$$

which is evaluated at $x = 0$. This energy contribution makes the determination of the vacuum difficult. The sign of the boundary constant C is not set, which could provide either a positive or negative contribution to the total energy of the system. This clearly shows that although the original choice for a vacuum should not be discarded, one should also consider other static solutions which in conjunction with the sign of C could provide a lower energy vacuum than before.

Apart from the original choice for a vacuum, one can consider static multi-soliton solutions. We restrict ourselves to one-soliton solutions since experience with similar models usually makes multi-soliton solutions unsuitable candidates.

When considering one-soliton solutions, one has the equations of the Bäcklund transformation (2.37) which are always true to simplify expressions. In particular we first consider the $\mathcal{H}_{bulk}$ term representing the energy in the bulk

$$\mathcal{H}_{bulk} = \int dx \left(\frac{|\partial_0 u|^2 + |\partial_1 u|^2}{1-uu^*} + m^2 uu^* \right) \ , \tag{3.42}$$

with $m = 2\sqrt{\beta}$. By direct substitution of the Bäcklund equations a simplified expression of the bulk energy is acquired

$$\mathcal{H}_{bulk} = \int dx (2m^2 uu^*) \ . \tag{3.43}$$

When the above expression is integrated throughout space, the result can be identified with the mass of the soliton solution u. However, now the integration is over the half line and specifically over the $[-\infty, 0]$ region.

The same equations can be used to express the $\mathcal{H}_B$ term of (3.41) . Specifically, the boundary constant C is determined by direct comparison of the Bäcklund equations of (2.37) and the boundary condition which appears in (3.31)

$$C = \frac{m}{2}\left(\delta\, e^{i(\theta+\Omega)} + \frac{1}{\delta}\, e^{-i(\theta+\Omega)}\right) . \tag{3.44}$$

with θ given by (2.39). At $x = 0$ and assuming that $V = 0$, the above expression simplifies to

$$C = \pm\frac{m}{\sqrt{1 + \tan^2(a)\coth^2(m\cos(a)x_0)}} . \tag{3.45}$$

This implies that $|C| \leq |m|$. This is the basic formula which relates the position of a static soliton x_0 with the charge parameter a and the boundary constant C. It is the necessary constraint for the static soliton to satisfy the boundary condition. Since both m and C are defined by the theory, the above relation is true only for specific choices of the soliton's position. Alternatively, one can think of this restriction emerging from the fact that for $|C| > |m|$, no choice of x_0 satisfies the boundary condition.

In the case where we choose $u = 0$ as a possible vacuum, the only remaining term in the total energy is

$$\mathcal{H}_{tot} = -2C . \tag{3.46}$$

Alternatively, one can consider a one-soliton solution where V is set to zero which appears in (2.65). In this case, both terms of (3.40) depend on the initial position of the soliton. However, after some calculations, the x_0 dependence drops out and the total energy is given by the expression

$$\mathcal{H}_{tot} = 2m\cos(a) . \tag{3.47}$$

It is far from obvious, which vacuum choice provides the minimum energy configuration. To determine this, one has to look at the expression of the boundary constant C in (3.45). This can be rewritten in the following form

$$y^2 + \frac{y^2 - y^4}{y^2 + F^2} = \cos^2(a) , \tag{3.48}$$

where

$$F^2 = \sinh^2(m\cos(a)x_0) \quad , \quad y = \frac{C}{m} \ . \tag{3.49}$$

In the above relation m and C should be treated as fixed parameters, while F can be varied through x_0. The left hand side of (3.48) is monotonically decreasing as F increases since $0 < y^2 < 1$. We observe the following

$$\begin{aligned} \cos^2(a) &= 1 \quad \text{when} \qquad F \to 0 \ , \\ \cos^2(a) &= y^2 \quad \text{when} \qquad F \to \pm\infty \ . \end{aligned} \tag{3.50}$$

This shows that moving the soliton away from the boundary decreases the energy of the system. On the extreme case where the soliton is placed at infinity, the model behaves as if no soliton exists, and the only contribution is the boundary term which coincides with the vacuum solution of $u = 0$. On the contrary as the soliton is placed closer to the boundary the energy increases. The maximum energy occurs when $F = 0$ at which point $\cos^2(a) = 1$ so that $H_{tot} = 2m$ which is greater than the energy $H_{tot} = -2C$ of the $u = 0$ vacuum.

Although the choice of vacuum in the bulk seems to be the most suitable choice in the boundary case too, one cannot rule out multi-soliton solutions that might provide lower energy configurations. This demands tedious calculations and remains as one of the open questions for this model.

3.2.2 Soliton reflections

In this section we investigate the reflection of solitons from the boundary. Mathematically this can be represented by a two-soliton solution satisfying the boundary condition. One of the solitons represents the incoming soliton whilst the other represents the reflected one. The point where the two solitons actually meet along the whole line as well as the phase shift due to their collision create an overall time-delay effect which can be calculated directly through the parameters of the scattering. This time-delay can be attributed to the interaction of the soliton with the boundary.

However, the most difficult step is to determine the restrictions that have to be imposed so that the two-soliton solution satisfies the boundary condition

$$\partial_1 u_{2s} = -C u_{2s}\sqrt{1 - u_{2s}u_{2s}^*} \ . \tag{3.51}$$

Energy and charge conservation laws demand that both the mass and the charge of the soliton are conserved by the boundary. This restricts the choice of the charge parameters a_1, a_2 to be either equal or opposite.

Due to the large expressions involved in the calculation, one is forced to expand both sides of the equation (3.51) to a Taylor series in exponentials of t, and match each term of the same order. Each term provides us with an equation involving the boundary parameter C. As mentioned in the previous section, the boundary constant has to be a real parameter. The real and imaginary parts of the equation yield two constraints on the parameters.

Let us consider this in more detail. We begin with a two-soliton solution, where the parameters are chosen in such a way so as to describe a soliton-soliton scattering. In this case, the charge parameters are taken to be opposite $a_1 = -a_2$ and the Bäcklund parameters to be $\delta_1 = -1/\delta_2$.

Furthermore, we adopt the following parametrisation which is more natural

$$\frac{K_1}{K_2} = e^{\varrho} \ , \ \frac{J_1}{J_2} = e^{i\zeta} \ , \ V = \tanh(\vartheta) \ , \tag{3.52}$$

where K_i and J_i are the shift parameters of (2.58). After both sides of the boundary equation are expanded as a Taylor series in time, we can discard the imaginary parts from all terms by using the following relation

$$\sin(\zeta) = -\tanh(\vartheta)\tan(a)\sinh(\varrho). \tag{3.53}$$

When the above equation is used the infinite set of equations collapse to a single constraint

$$C = m\cos(a)\cosh(\vartheta)\left(\frac{\cos(\zeta) + \cosh(\varrho)}{\sinh(\varrho)}\right) \ . \tag{3.54}$$

When the shift parameters are fixed according to the above relations, the two-soliton solution satisfies the boundary condition and this process describes a soliton being reflected by the boundary.

The fact that only relative shifts in both normal and internal $U(1)$ space are important should be expected from time translational and $U(1)$ invariance of the model. The non-topological solitons in the CSG theory are reflected as solitons carrying the same charge Q. This is because the boundary potential does not breach the $U(1)$ symmetry since it depends only on $|u|$.

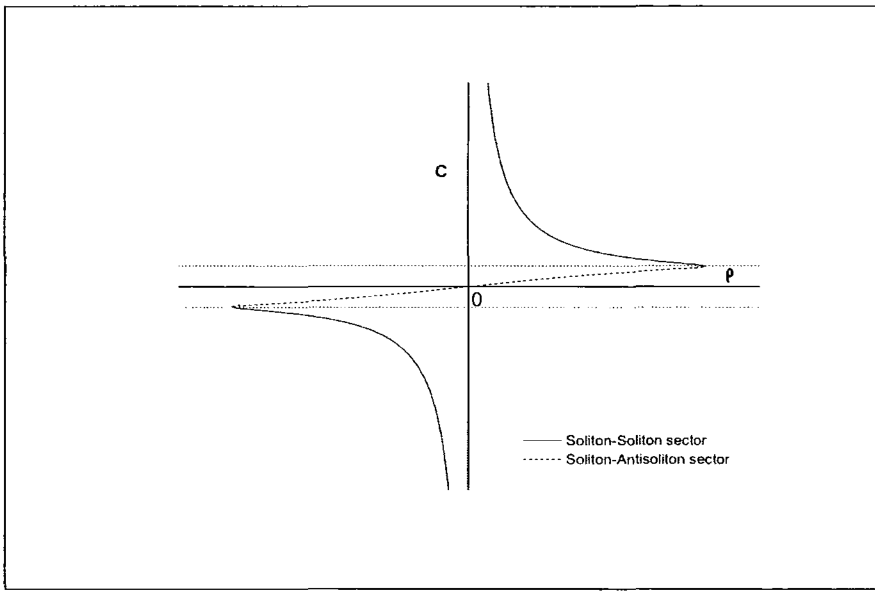

Figure 3.1: C in terms of ϱ

We shall consider a and ϑ as fixed parameters and use equations (3.53) and (3.54) to determine the parameter ϱ in terms of C. By eliminating the parameter ζ, one ends up with the following quadratic equation for C

$$C^2 - 2\,\frac{Cm\cos(a)\cosh(\vartheta)\cosh(\varrho)}{\sinh(\varrho)} + \left(\sinh(\vartheta)^2 + \cos(a)^2\right)m^2 = 0 \ . \tag{3.55}$$

The solutions of the above equation can be plotted to present the dependence on ϱ. The plot involves two branches (Fig. 3.1) due to the sign ambiguity, which are mutually exclusive. The plot shows that a soliton can always be reflected by the boundary. The branches meet at the points

$$C = \pm m\sqrt{\cos^2(a) + \sinh^2(\vartheta)} \ , \quad \coth(\varrho) = \frac{\sqrt{\sinh^2(\vartheta) + \cos^2(a)}}{\cos(a)\cosh(\vartheta)} \ . \tag{3.56}$$

In the limit $a \to 0$ the two branches of the plot can be identified with the soliton-soliton and soliton-antisoliton sector of the reflection process at the sine-Gordon limit (Fig. 3.2). For fixed values of ϑ and $a = 0$, it is the value of the boundary constant C which determines whether a soliton is reflected as a soliton or an antisoliton. For C small, a soliton is reflected as an antisoliton (Neumann boundary conditions for $C = 0$), while for C large a soliton is reflected as a soliton (Dirichlet boundary conditions for $C = \infty$). For $C = m\cosh(\vartheta)$ the branches do not meet as in the CSG case. This specific value of C corresponds to a logarithmic divergence that appears in the classic time delay for the sine-Gordon case. These results coincide with the results derived by previous treatments of the boundary sine-Gordon model [19].

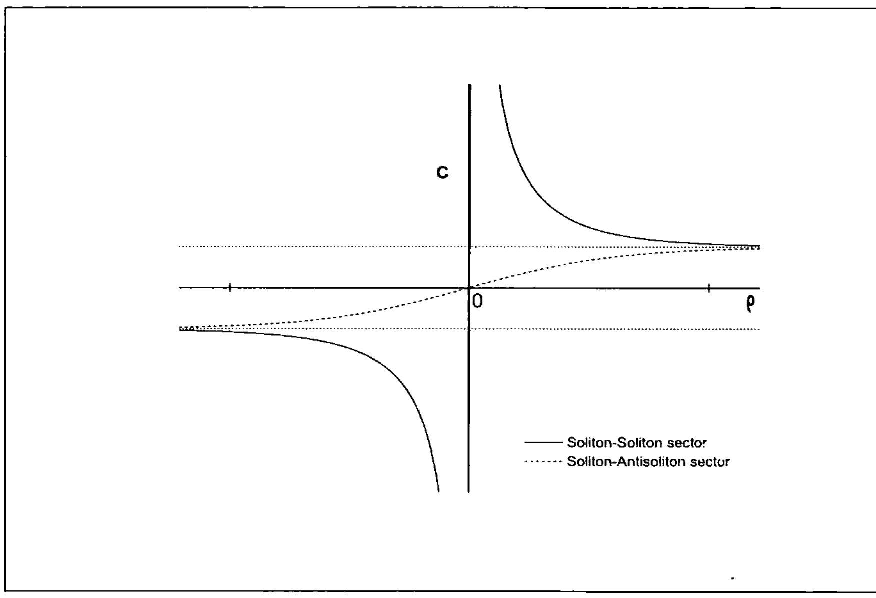

Figure 3.2: C in terms of ϱ for chargeless case

3.2.3 The classical time delay

The time delay which appears at the scattering of a soliton off the boundary, can be calculated directly from the asymptotic values of the solution at $t = \pm\infty$. We begin with the two-soliton solution and change to a frame of reference which moves with the incoming soliton (i.e. $x = Vt$. In the limit $t = -\infty$ the solution becomes

$$S_- = \lim_{t\to-\infty} u_{2s} = \frac{\cos(a)\ e^{i(A_1+B_1)}}{\cosh(P(-x+x_1)+r)} \ , \tag{3.57}$$

where

$$A_1 = \frac{P\sin(a)}{\cos(a)}\left[(1-V^2)t - Vx - y_1\right] \ , \ \tan(B_1) = -\frac{V\sin(a)}{\cos a} \tag{3.58}$$

and

$$P = \frac{m\cos(a)}{\sqrt{1-V^2}} \quad , \tag{3.59}$$

$$r = \frac{1}{2}\ln\frac{V^2}{\cos^2(a) + V^2\sin^2(a)} \quad . \tag{3.60}$$

The parameters x_i and y_i represent regular shifts that were introduced in (2.58). The solution, as expected, describes a single incoming soliton at early time far away from the boundary.

We repeat the same calculation, but now we change to the frame of reference of the outgoing soliton (i.e. $x = -Vt$) and calculate the limit of the two-soliton solution at $t = +\infty$ which yields

$$S_+ = \lim_{t=+\infty} u_{2s} = \frac{\cos(a)\; e^{i(A_2+B_2+\pi)}}{\cosh(P(-x-x_2)+q)} \quad , \tag{3.61}$$

where

$$A_2 = \frac{P\sin(a)}{\cos(a)}\left[(1-V^2)t + Vx - y_2\right] \quad , \quad \tan(B_2) = \frac{V\sin(a)}{\cos(a)} \quad , \tag{3.62}$$

and

$$q = \frac{1}{2}\ln\frac{V^2}{\cos^2(a) + V^2\sin^2(a)} \quad . \tag{3.63}$$

Once again this is a single soliton solution representing the reflected soliton far away from the boundary wall.

The asymptotic solutions S_+, S_- contain all the information needed to calculate the time-delay. The latter is a combination of two separate events. Firstly, a phase shift is induced during the scattering of the two solitons. Before the two solitons re-emerge as two separate entities, the reconfiguration of the solution creates a phase shift which is equivalent to a time delay. Secondly, the centre of mass of the two-soliton solution does not necessarily lie at the boundary. This implies that the two solitons actually meet at a different point than $x = 0$. This creates again a time delay which may be either positive or negative corresponding to an attractive or repulsive boundary potential respectively.

Ignoring any interaction between the two solitons, one can project the trajectories of S_+ and S_- on $x - t$ diagram and find the point where these cross (Fig. 3.3). The distance of this point from the boundary is proportional to the time delay $\Delta\tau$ which

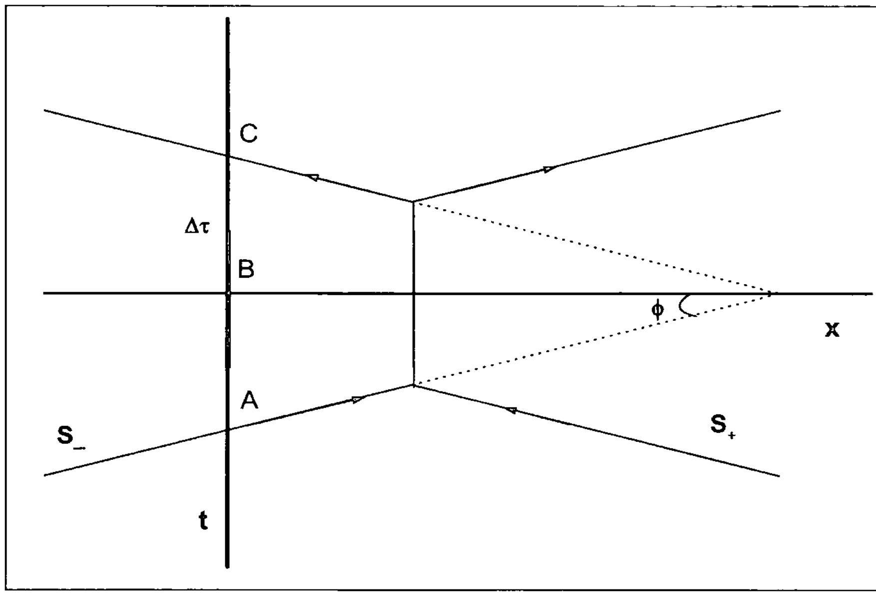

Figure 3.3: Scattering diagram

in the diagram is given by the distance (AC). The time delay corresponds to the time interval in which the soliton appears to be absorbed by the boundary before it reemerges as a well defined entity . The two solitons move across the following lines

$$\begin{aligned} S_- : \quad t &= \frac{1}{V}\left(P(-x+x_1)+r\right) \quad , \\ S_+ : \quad t &= \frac{1}{V}\left(P(x+x_2)-q\right) \quad , \end{aligned}$$

as dictated by (3.61) and (3.57). The lines cross at

$$x_0 = \frac{1}{2}\left(x_1 - x_2 + \frac{r+q}{P}\right) = \frac{x_1-x_2}{2} + \frac{r}{P} \quad , \tag{3.64}$$

since $r = q$. The time delay is finally

$$\Delta\tau_{CSG} = \frac{2x_0}{V} = \frac{(x_1-x_2)}{V} + \frac{\sqrt{1-V^2}}{mV\cos(a)}\ln\left(\frac{V^2}{\cos^2(a)+V^2\sin^2(a)}\right) \quad . \tag{3.65}$$

In the expression above, the first term of the right-hand side represents the time delay caused by the non-symmetric character of the solution with respect to the boundary. In the special case where $x_1 = x_2$, the centre of mass lies on the boundary and the term vanishes. The second term is independent of the initial position of the two solitons or the boundary potential and is caused by the phase shift of the scattering process.

The relative position of the two solitons is however fixed according to the constraint equations (3.53) and (3.54) which ensure that solution satisfies the boundary condition. Specifically the parameter ϱ corresponds exactly to the $x_1 - x_2$ difference up to the overall factor P. It is thus possible to express the time delay in terms of the boundary constant, by solving the constraint equations and substituting the relative position of the solitons. We choose to express the velocity parameter V in terms of the rapidity ϑ for simplicity reasons. After a few straightforward calculations we recover the following expression for the time delay

$$\Delta\tau_{CSG} = \frac{\ln Q}{2m\cos(a)\sinh(\vartheta)} \tag{3.66}$$

where

$$Q = \frac{\sinh^4\vartheta\left((\cos^2(a)+\sinh^2(\vartheta))m^2 + 2Cm\cos(a)\cosh(\vartheta)+C^2\right)}{(\cos(a)^2+\sinh^2(\vartheta))^2\left((\cos^2(a)+\sinh^2(\vartheta))m^2 - 2Cm\cos(a)\cosh(\vartheta)+C^2\right)} \tag{3.67}$$

In the special limit of $a = 0$, the time delay for the sine-Gordon model is recovered

$$\Delta\tau_{SG} = \frac{1}{m\sinh(\vartheta)}\ln\left(\tanh^2(\vartheta)\frac{m\cosh(\vartheta)+C}{m\cosh(\vartheta)-C}\right) \quad . \tag{3.68}$$

This is exactly the time delay calculated for the sine-Gordon theory in the presence of a boundary [19], but for the restricted class of boundary conditions which admit $\phi = 0$ as a vacuum.

3.2.4 Boundary bound states.

In this section we examine the spectrum of bound states. Once again, for the the boundary condition to be satisfied we need to introduce restriction to some of the parameters as in the soliton reflections.

The simplest bound state that we can have is the static single soliton that was introduced in (2.65). The solution is not really static, as the imaginary phase survives the setting of the speed parameter V to zero. The solution is static only in the sense that the centre of mass doesn't translate in the x direction, although the wave oscillates with fixed angular velocity $\omega = m\sin(a)$.

When a boundary is introduced, the static soliton, can satisfy the boundary condition for $|C| \leq |m|$, when its position is fixed according to equation (3.45). At the chargeless limit any time dependence vanishes and the solution collapses to a static single soliton of the sine-Gordon theory, fixed at the boundary.

Breathers that have been constructed by the method described in section (2.6.2), can also be shown to satisfy the boundary condition. The demand that C is real still holds. However, all the arbitrary phase shifts are now real numbers and constrained. We examine breather solutions that emerge from the soliton-soliton case. Just as before, the solution does satisfy the boundary condition with some restrictions involving the arbitrary parameters. Once more a Taylor expansion of the boundary equation is needed. The parametrisation used in this case is

$$K_1 = e^{\varrho} \quad , \quad J_1 = e^{\zeta} \quad , \quad V = \tan(\vartheta) \quad , \tag{3.69}$$

while the parameters K_2 and J_2 have been been properly fixed so that this is a breather solution. The first restriction needed for the solution to satisfy the boundary condition is

$$\sinh(2\zeta) = -\tan(a)\tan(\vartheta)\sinh(2\varrho) \quad . \tag{3.70}$$

The parameter ϑ plays the role of the rapidity, which has now been analytically continued. The second restriction which completes the necessary requirements for a boundary bound state is

$$C = m\cos(a)\cos(\vartheta)\left(\frac{\cosh(2\zeta) - \cosh(2\varrho)}{\sinh(2\varrho)}\right) \tag{3.71}$$

Both relations can be recovered by analytical continuation of the corresponding relations of (3.53) and (3.54) after the necessary restrictions for a breather solution have been already taken into account. It is instructive to examine the relation between the parameters C and ϱ (Fig. 3.4) since it provides valuable insight to the structure of bound states. It is straightforward that there are two distinct regions of values of C that do not correspond to any bound state. This regions are defined by the limit values

$$C = \pm m\left(\cos(a)\cosh(\theta) \pm \sin(a)\sinh(\theta)\right) \quad . \tag{3.72}$$

At the chargeless limit, the regions collapse to the single values

$$C = \pm m \cosh\theta \quad , \tag{3.73}$$

which coincided with the logarithmic divergence appearing in the time delay for the soliton reflection.

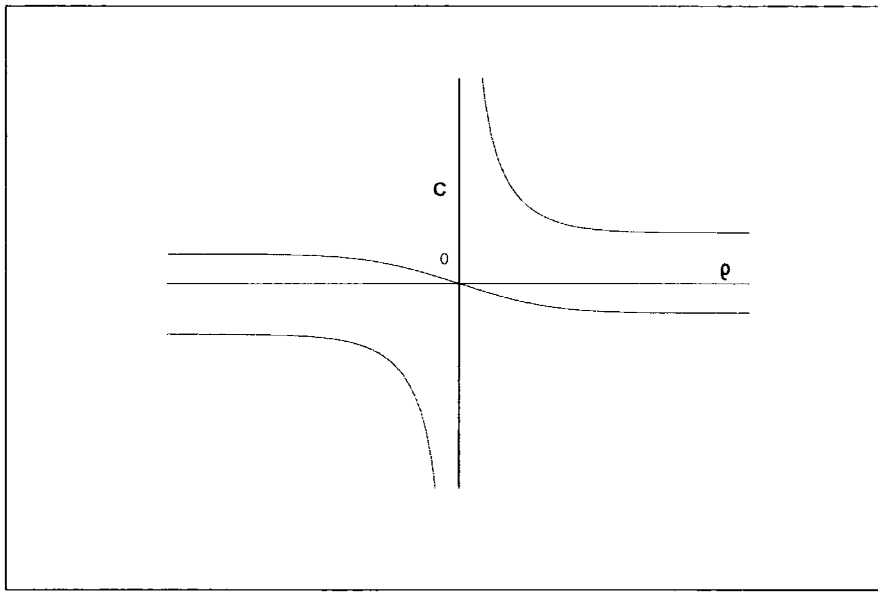

Figure 3.4: C in terms of ϱ for Breather case

3.2.5 Particle Reflections.

In this section we consider the spectrum of particles and their reflection factors in the presence of a boundary. When small perturbations around the vacuum are considered

$$u = 0 + \epsilon(x,t) \ , \tag{3.74}$$

the theory becomes linear when higher order terms in ϵ are ignored

$$(\partial_0^2 - \partial_1^2)\epsilon(x,t) + m^2\epsilon(x,t) = 0, \tag{3.75}$$

where $m^2 = 4\beta$. The solution to the above equation is the familiar plane waves solution

$$\epsilon(x,t) = e^{-i\omega t}\left(Ae^{ikx} + Be^{-ikx}\right), \tag{3.76}$$

where k and ω are related through

$$\omega^2 = k^2 + m^2. \tag{3.77}$$

For small fluctuations around the vacuum $u = 0$, the boundary condition (3.31) becomes

$$\partial_1\epsilon(x,t) = -C\epsilon(x,t). \tag{3.78}$$

In order to calculate the reflection factor when particles bounce of the boundary wall, we substitute in the last relation the particle solutions (3.76). The constant A of the right propagating waves is taken to be one, since it has to do with the characteristics of the particle beam. The reflection factor is identified with the constant B, which corresponds to a phase change as the particles encounter the boundary

$$B = \frac{ik + C}{ik - C} . \tag{3.79}$$

The reflection factor, as expected, depends on C which as stated before, appears as a free, real parameter in the boundary condition. For $C = 0$, the reflection factor is equal to $B = 1$ and no phase appears between the two waves upon their scattering off the boundary. This is consistent with the fact that the boundary term is proportional to the boundary constant, so when C is set to zero, the boundary term vanishes.

Particle solutions can be related to bound states through the pole appearing in B. Indeed one may choose $k = -iC$ and apply this to a solution of the form $\frac{1}{B}\epsilon(x,t)$. The remaining terms depend explicitly on the boundary constant

$$\epsilon(x,t) = e^{-i\left(\sqrt{m^2-C^2}t+-iCx\right)} . \tag{3.80}$$

This solution is square integrable only for a specific range of values for the boundary constant. Specifically if C is positive then the solution is not square integrable since it is exponentially increasing as $x \to \infty$.

When $-m < C < 0$, then $\epsilon(x,t)$ represents a square integrable exponentially decreasing solution as $x \to \infty$. It oscillates with constant angular velocity $\omega = \sqrt{m^2 - C^2}$ and is therefore a stable bound state. It can also be viewed as the tail of a static one-soliton solution satisfying the boundary condition, with the parameters adjusted in such a way its centre of mass goes to positive infinity. Examining the condition for the static soliton to obey the boundary condition, this limit can be achieved as $x_0 \to \infty$, i.e. we must take the charge is such a way that $C = -m\cos a$.

Finally in the region $C < -m$, the solution can increase exponentially in time. This shows that the vacuum solution $u = 0$ is no longer stable. In fact the particle behaviour which corresponds to a small perturbation around the vacuum seems to be ever increasing. This instability can be understood through a rather impressive mechanism in which a chargeless soliton is emitted from the boundary, effectively changing the value of C so that $u = 0$ is now stable.

Recall from section (2.6.1) that for a chargeless soliton we should take the opposite sign for $\sqrt{1-uu^*}$ on each side of the centre of the soliton where $|u| = 1$. The instability can be viewed as a left moving chargeless soliton which approaches the boundary from $x = \infty$. In the beginning while the soliton is far away from the boundary $u = 0$ so that the boundary potential of (3.41) is $H_B = -2C$. As the centre of the soliton passes through $x = 0$, the sign of the square root in the boundary potential changes. As the soliton moves to $x = -\infty$, u returns to 0 near the boundary but now we take the boundary energy with the opposite sign $H_B = 2C$. Effectively the sign of C has been flipped to a positive value. The energy released from the boundary is $4C > 4m$, which is greater than the rest mass of a single chargeless soliton. At $C = -m$, the soliton is emitted with infinitesimal velocity. As C decreases, more energy is given up by the boundary and the soliton can be emitted with larger V.

This process agrees with the infinite time-delay effect which was encountered in the soliton reflections section. The soliton emission represents the time reversal picture of that effect in the chargeless soliton case (Fig. 3.2).

3.3 Discussion

In this chapter we introduced a boundary term in the complex sine-Gordon Lagrangian and demanded that the system remains integrable. First we constructed some of the infinity of conserved quantities of the theory and derived suitable boundary conditions which preserve integrability. In the presence of a boundary, we examined the vacuum structure and showed that the lowest energy configuration in the bulk remained still the most suitable candidate for a vacuum.

Soliton reflections off the boundary were also studied and the necessary constraint equations were written down in terms of the phase shift parameters. The set of equations was derived by demanding that the two-soliton solution satisfies the boundary condition. Moreover the time delay induced by the scattering process was calculated in terms of the boundary constant C and was found to coincide in the chargeless limit with the time delay of the sine-Gordon theory.

Finally we examined bound states and found the necessary restrictions for the boundary condition to be satisfied. A special relation was shown to exist, identifying bound states as the asymptotic part of soliton solutions.

An obvious extension of the results so far is to consider the quantum case of the boundary CSG model. As the simplest case of a class of generalisations of the sine-Gordon theory, the results are vital to the proper understanding of more complicated models. The following chapter addresses the problem of the quantum case and attempts to build the whole spectrum of states using two different techniques.

Chapter 4

Quantum complex sine-Gordon model on a half line

4.1 Introduction

In the previous chapters we have studied the classical complex sine-Gordon theory in the bulk and on a half line. We have written down analytic solutions, examined their behaviour and studied their interaction with the boundary. Having established a firm understanding of the classical picture we shall attempt to examine the corresponding quantum theory.

The chapter begins with the quantum case in the bulk which has already been studied in a number of papers [46, 53, 58]. After a brief review of the most important results in the bulk, we shall concentrate on the half-line case.

A brief description of the two different methods that will be used to render the mass spectrum of boundary-bound states was presented in the introduction.

The first method is the semi-classical stationary-phase approach which is the field theory analogue of the WKB method of quantum mechanics. A generalised version of Bohr-Sommerfeld quantisation will be used to obtain the semi-classical spectrum and its first order corrections. As we shall see, these corrections induce only a finite renormalisation of the coupling constant in the same fashion as in the bulk case.

Following that, the bootstrap programme devised by Ghoshal and Zamolodchikov [6] will be used to construct the quantum reflection factors of soliton states and use their poles to explore the existence of boundary-bound states. This will provide us with an exact spectrum of states which will be directly compared with the one of the previous section.

Both methods have a quite different starting point but since both describe the same physical setup, we expect the results to coincide. The energy spectrum derived by each method will prove to be the same proving the validity of our results.

The chapter finishes with a few general remarks about the results of the quantum case including a brief discussion about the poles in the bootstrap method and the necessary conditions in order for them to lie within the physical strip.

4.2 Quantum complex sine-Gordon model in the bulk

The complex sine-Gordon model as a quantum field theory in the bulk was studied relatively soon after the model's introduction. The classical treatment which showed the theory to be completely integrable and to possess soliton solutions carrying a $U(1)$ charge, prompted researchers to look into the quantum case in the hope that the nice features of the model persisted in this limit too.

The investigation of the quantum case began with the work of de Vega and Maillet [59] in which they showed that the S-matrix is factorisable at tree level. The model remains integrable and continues to accept soliton solutions in the quantum case. Provided that a specific counterterm which depends on the field is added to the Lagrangian, the S-matrix is also factorisable at one-loop level. In their following paper [46] they used the inverse scattering method to obtain the classical two-soliton solution and the spectrum of states using the semi-classical methods by Dashen, Hasslacher and Neveu [3, 4].

The two-loop order case was studied by Bonneau in [58] who continuing down the path of de Vega and Maillet showed that the theory is non-renormalisable unless a finite number of counterterms (quantum corrections) are added.

After a gap of almost ten years the quantum complex sine-Gordon case was revisited by Dorey and Hollowood [53] in the light of the theory emerging as a gauged WZW model [41]. With the semi-classical results of de Vega and Maillet as a guide, they proposed an exact S-matrix based on the demands of the bootstrap programme.

Before reviewing the relevant results in the literature, we need to reinstate the coupling constant ξ into all expressions. The coupling constant was dropped in the first chapter since it appeared as a total factor in front of the Lagrangian after suitably rescalling the fields. In the quantum case however, the coupling constant acts much in the same way as Planck's constant $\hbar$, and thus cannot be ignored. Since ξ is no longer set to unity all solutions now depend on the coupling constant as dictated by the transformation in (2.12). Moreover, one has to bare in mind that the Lagrangian of the model comes with an overall factor of $\frac{1}{\xi^2}$ which should appear in expressions like the energy and charge.

In their paper Dorey and Hollowood argued that it is necessary to only consider specific values for the coupling constant. Specifically they argued that the only acceptable values are

$$\xi^2 = \frac{4\pi}{k} \quad , \quad k > 1 \ , \tag{4.1}$$

where k is an integer. This agrees with the WZW interpretation of the theory where k corresponds to the level of the $SU(2)/U(1)$ coset model. The quantum theory of this model is well defined only if the level is an integer greater than one. This also is consistent with the description of parafermions since it restricts the spectrum only to a finite number of states. With this constraint their proposed S-matrix reproduces the semi-classical spectrum of states derived by de Vega and Maillet. In addition they proposed that the charge is conserved if it is defined modulo k, which agrees with the statement made in the second chapter (Sec. 2.5). In the following, we shall use k instead of the coupling constant ξ for reasons of simplicity.

The quantum spectrum can be found by using the Bohr-Sommerfeld quantisation rule

$$S(u) + M(u)\tau = 2\pi n \quad , \quad n \in \mathbb{Z} \ , \tag{4.2}$$

where S is the action functional, M the mass and τ the period of the solution u. Since no topological distinction exists between the vacuum and the soliton sector, the soliton may be regarded as the the basic particle of the CSG theory. We shall use the static one-soliton solution of (2.65). As we have already seen this solution does not translate in the x direction but oscillates in a breather-like fashion. As pointed out by Ventura and Marques [60], and Montonen [61], the Bohr-Sommerfeld quantisation is equal to charge quantisation for scalar field theories enjoying a global $U(1)$ symmetry. For the CSG case the only time dependence for the soliton solution in the rest frame is restricted to the phase i.e. uu^* does not depend on time. It is easy to show that for the static one-soliton

$$S(u) + M(u)\tau = 2\pi Q \ , \tag{4.3}$$

which in turn implies

$$Q = ne \ , \tag{4.4}$$

with e the basic charge that we shall set to $e = 1$. This corresponds to a tower of states with ever increasing charge. However in the classical case the charge is a periodic function (Fig. 2.1). As suggested by Dorey and Hollowood the charge should be defined modulo k which leads to a finite spectrum depending on k

$$Q = \pm 1, \pm 2, ..., \pm \tfrac{k-1}{2} \qquad \text{for } k \text{ odd} , \tag{4.5}$$

$$Q = \pm 1, \pm 2, ..., \pm \tfrac{k-2}{2}, \tfrac{k}{2} \qquad \text{for } k \text{ even} . \tag{4.6}$$

From the classical expression for the charge (2.42) we have

$$Q = \frac{k}{\pi}(\frac{\pi}{2} - a) . \tag{4.7}$$

The quantisation of the charge is in fact the quantisation of a

$$2\pi Q = 2\pi n \Leftrightarrow a = \frac{\pi}{k}\left(\frac{k}{2} - n\right) . \tag{4.8}$$

Through the quantisation of the charge parameter, the energy spectrum

$$M = \frac{k}{\pi} m \cos(a) , \tag{4.9}$$

is also obtained

$$M(Q) = \frac{km}{\pi}\left| \sin\left(\frac{\pi Q}{k}\right)\right| . \tag{4.10}$$

The Bohr-Sommerfeld quantisation provides us with the quantum spectrum only up to leading order. According to de Vega and Maillet the next order correction is achieved by a simple renormalisation of the coupling constant

$$\xi^2 \to \xi_R^2 = \frac{\xi^2}{1 - \xi^2/4\pi} . \tag{4.11}$$

or equivalently

$$k \to k_R = k - 1 . \tag{4.12}$$

In this level an S-matrix may be written down that reproduces the semi-classical spectrum to leading order. In their paper Dorey and Hollowood first presented a minimal choice for the meson-soliton scattering matrix

$$S_{1Q} = F_{Q-1}(\theta) F_{Q+1}(\theta) . \tag{4.13}$$

which reproduced the semi-classical behaviour and agreed with the results of deVega and Maillet. Moreover, through this they confirmed that the meson in the CSG theory can be identified with the $Q = 1$ soliton. The function $F_Q(\theta)$ is defined as

$$F_Q(\theta) = \frac{\sinh\left(\frac{\theta}{2} + \frac{i\pi}{2k}Q\right)}{\sinh\left(\frac{\theta}{2} - \frac{i\pi}{2k}Q\right)} . \tag{4.14}$$

S-matrices constructed from products of the $F_Q(\theta))$ automatically satisfy unitarity and analyticity constraints. With the above result as a starting point they proposed the following S-matrix for arbitrary charge

$$S_{Q_1 Q_2} = F_{Q_1-Q_2}\left[\prod_{n=1}^{Q_2-1} F_{Q_1-Q_2+2n}\right]^2 F_{Q_1+Q_2}, \tag{4.15}$$

which satisfies all the familiar restrictions and has the correct pole structure. In addition they pointed out that this is exactly the minimal S-matrix associated with the Lie algebra a_{k-1} and conjectured that for the specific choices of the coupling this S-matrix should be exact.

With the general form of the S-matrix the review of the bulk case is concluded. More details on the above results may be found in the relevant papers. Nevertheless, this presentation contain all the ingredients needed to examine the half-line case. We shall begin with the semi-classical approach as described in the introduction.

4.3 Semi-classical quantisation

In this section we shall attempt to build the complete spectrum of quantum boundary-bound states for the complex sine-Gordon model on a half line. The fact that the model possesses exact periodic solutions makes the stationary-phase method the most suitable candidate for the quantisation. As pointed out in the introduction, this is a generalised version of the Bohr-Sommerfeld quantisation

$$S_{qu} - T\frac{\partial S_{qu}}{\partial T} = 2\pi n , \tag{4.16}$$

where $S_{qu} = S_{cl} - \Delta$. We shall use the ordinary quantisation condition for the classical action S_{cl} and then we shall calculate the quantum corrections factor of (1.49).

The boundary term of (3.39) preserves the $U(1)$ charge as it only depends on mod(u). The theory remains $U(1)$ invariant and therefore the Bohr-Sommerfeld quantisation is equivalent to charge quantisation on the half line case as it was for the bulk. The quantisation condition for the classical action reads

$$S_{cl}[u_{cl}] + E_{cl}[u_{cl}]T = 2\pi Q = 2\pi n \ , \tag{4.17}$$

with u_{cl} the static one soliton solution of (2.65). In the previous chapter we saw that this solution satisfies the boundary condition. It is a periodic solution with period T exhibiting a breather-like behaviour. It provides the perfect starting point as the simplest boundary bound state of the classical theory. The position of the centre of mass will determine the charge of the bound state as a fraction of the soliton charge in the bulk. We can calculate the charge of the static soliton through the expression

$$Q = -i \int_{-\infty}^{0} dx \frac{u^* \partial_0 u - u \partial_0 u^*}{1 - uu^*} \ . \tag{4.18}$$

This time however the integration takes place on the half line and finally yields

$$Q = \frac{k}{2\pi} (\pi - b - a) \ . \tag{4.19}$$

The charge now depends on the boundary parameter $C = \cos(b)$, which enters the calculation through the position of the centre of mass. To check that this formula is correct, we can set $b = \frac{\pi}{2}$ which implies that we have Neumann boundary conditions thus placing the soliton at exactly $x = 0$. The charge is then exactly half of the equivalent charge of the soliton in the bulk as expected. The quantisation condition now reads

$$Q = \frac{k}{2\pi} (\pi - b - a) = n \ , \tag{4.20}$$

or in terms of the charge parameter

$$a = \pi - b - \frac{2\pi n}{k} \ . \tag{4.21}$$

The quantisation of a provides us with the first approximation of the semi-classical spectrum

$$E_n = \frac{km}{2\pi} |\cos(a)| = \frac{km}{2\pi} \left| \cos(\pi - b - \frac{2\pi n}{k}) \right| \ . \tag{4.22}$$

Having a general formula for the energy, it is useful to calculate the energy difference between two adjacent states. We shall use this in the following section where we shall compare it with the corresponding bootstrap result. For simplicity reasons, we shall assume that the values of the parameters are such that the cosine is positive for both states as is their difference so that we can ignore any modulus appearing. The energy difference is then written

$$E_{n+1} - E_n = \frac{km}{2\pi}\left(\cos(\pi - b - \frac{2\pi(n+1)}{k}) - \cos(\pi - b - \frac{2\pi n}{k})\right) , \qquad (4.23)$$

which after some manipulation simplifies to

$$E_{n+1} - E_n = \frac{km}{\pi}\cos\left(\frac{\pi}{2} - \frac{\pi}{k}\right)\cos\left(\frac{\pi(2n+1)}{k} - \frac{\pi}{2} + b)\right) . \qquad (4.24)$$

We shall return to this result at the end of the next section, after we have obtained a relevant expression from the bootstrap programme.

The following step consists of calculating the quantity Δ of (1.44). We first calculate the sum over the stability angles, while the counter terms will be introduced later to make the result finite. In order to obtain all possible states one has to consider the theory in a box. The theory is already bounded on the right from the original boundary, therefore another boundary should be introduced restricting the system in the finite volume $[-L, 0]$thus forcing all energy levels to become discrete. We can afterwards take the limit $L \to -\infty$ to recover the original system. The most appropriate choice would be Dirichlet boundary conditions $u = 0$, which reproduce the correct behaviour of u at infinity.

The stability angles are obtained by solving the linearised stability equation (1.39) for a given classical solution. In our case we perturb around the static one-soliton solution. Once the solutions $\chi(x,t)$ for the stability equation are found then the stability angles can be calculated from (1.40). Instead of solving directly the stability equation we can follow the method of Corrigan and Delius to calculate the sum of the stability angles through the reflection factors. We begin with the classical two-soliton solution of the CSG model which satisfies the stability equation. By fixing the free parameters we can make one of the solitons S_1 static by taking $\delta_1 = 1$ and the other S_2 very small by taking the charge parameter a_2 close to $\frac{\pi}{2}$. Effectively we are left with a small perturbation around a static one-soliton background. This is exactly the same method that Dashen, Hasslacher and Neveu followed to calculate the stability angles of the sine-Gordon model. At infinity the static soliton is practically zero whilst the perturbation appears as plane waves

$$\chi(x,t) = e^{-i\omega t}(e^{ik_s x} + R_s e^{-ik_s x}) . \qquad (4.25)$$

The reflection factor is found to be

$$R_s = -\frac{(\cos(b - i\theta))}{(\cos(b + i\theta))} \frac{(\sin(a + i\theta) - 1)}{(\sin(a - i\theta) - 1)} , \tag{4.26}$$

where we have set $m = 1$ without loss of generality. The rapidity θ which has been used is related to the momentum through $k = \sinh(\theta)$. The parameters a and b are the familiar charge and boundary parameters. From the above expression for $\chi(x, t)$ and (1.40) we can substitute

$$\Delta = \frac{1}{2} \sum_i v_i = \frac{T}{2} \sum_i \omega_i , \tag{4.27}$$

where $\omega_i^2 = k_i^2 + 1$ and the counter terms have been neglected. Placing the system in a box allows for discrete values of the wave-number k which should not be confused with the coupling constant. From Δ we must now subtract the vacuum contributions

$$\Delta = \frac{T}{2} \sum_i \sqrt{k_{s,i}^2 + 1} - \sqrt{k_{0,i}^2 + 1} . \tag{4.28}$$

From the Dirichlet boundary conditions we can obtain the following equation relating the discrete momenta with the reflection factors

$$e^{-2ik_s L} = \frac{(\cos(b - i\theta))}{(\cos(b + i\theta))} \frac{(\sin(a + i\theta) - 1)}{(\sin(a - i\theta) - 1)} . \tag{4.29}$$

A similar equation exists for the fluctuations around the vacuum

$$e^{-2ik_0 L} = -\frac{i \sinh(\theta) + \cos(b)}{i \sinh(\theta) - \cos(b)} . \tag{4.30}$$

Using the same argument as Corrigan and Delius, we can define a function $\kappa(k_0)$ such that for large k we may write

$$k_s = k_0 + \frac{\kappa(k_0)}{L} , \tag{4.31}$$

where the index i has been suppressed. This is possible since in the limit $\theta \to +\infty$ and taking into account the general quantisation condition of (4.20), both reflection factors R_s and R_0 are equal. Through the difference $k_s - k_0$ a function $\kappa(\theta)$ may be defined using the ratio of the corresponding reflection factors

$$e^{-2i\kappa(\theta)} = -\frac{(\sin(a + i\theta) - 1)}{(\sin(a - i\theta) - 1)} \frac{(\sin(b + i\theta) - 1)}{(\sin(b - i\theta) + 1)} . \tag{4.32}$$

We can now calculate Δ in terms of κ. We can substitute (4.31) in (4.28) and then expand the expression in terms of L. Keeping only the leading term of the expansion we end up with

$$\Delta \sim \frac{T}{2L} \sum_i \frac{k_i^{(0)} \kappa(k_i^{(0)})}{\sqrt{(k_i^{(0)})^2 + 1}} , \tag{4.33}$$

which in the limit $L \to +\infty$ can be substituted with the integral form

$$\Delta = \frac{T}{2\pi} \int_0^{+\infty} dk \frac{k}{\sqrt{k^2+1}} \kappa(k) . \tag{4.34}$$

The calculation is greatly simplified if we change variables to θ

$$\Delta = \frac{T}{2\pi} \int_0^{+\infty} d\theta \sinh(\theta) \kappa(\theta) . \tag{4.35}$$

The integral is divergent but we can introduce specially chosen counter-terms to obtain a finite result. We begin with an integration by parts

$$\Delta = \frac{T}{2\pi} \left([\kappa \cosh(\theta)]_0^{+\infty} - \int_0^{+\infty} \cosh(\theta) \frac{d\kappa}{d\theta} \right) . \tag{4.36}$$

The first term is not divergent since the function κ approaches zero (through the quantisation condition) as θ goes to infinity. In this limit the combination $\kappa(\theta)\cosh(\theta)$ is zero. In addition, $\kappa(0) = 0$ so the first term in (4.36) vanishes. The second term is however divergent and counter terms have to be introduced to cancel infinities. The latter appear in the same fashion as the logarithmic divergencies in the bulk which are tackled through normal ordering. With some straightforward manipulation the derivative term yields

$$\frac{d\kappa}{d\theta} = -\frac{\cos(b)}{\cosh(\theta) - \sin(b)} + \frac{\cos(a)}{\cosh(\theta) - \sin(a)} . \tag{4.37}$$

The two terms are almost identical and both divergent. A logical choice of counter-terms seems to be

$$-\frac{\cos(b)}{\cosh(\theta)+1} + \frac{\cos(a)}{\cosh(\theta)+1} , \tag{4.38}$$

where the first removes the divergence associated with the boundary and the second with the one in the bulk. The complete expression to be calculated is now

$$\begin{aligned}\Delta \;=\; & -\int_0^{+\infty} d\theta \cosh(\theta) \\ & \left(\frac{\cos(a)}{\cosh(\theta) - \sin(a)} - \frac{\cos(a)}{\cosh(\theta)+1} - \frac{\cos(b)}{\cosh(\theta) - \sin(b)} + \frac{\cos(b)}{\cosh(\theta)+1} \right) ,\end{aligned}$$

which finally yields

$$\Delta = \frac{T}{2\pi}\left(-\cos(a) + \cos(b) + b\sin(b) + \frac{\pi}{2}\sin(b) - a\sin(a) - \frac{\pi}{2}\sin(a)\right) . \quad (4.39)$$

Having determined the form of Δ we are now in a position to calculate the corrections to the classical Bohr-Sommerfeld rule. The expression of Δ depends on the period T through the charge parameter a

$$\sin(a) = \frac{2\pi}{T} . \quad (4.40)$$

The correction term of (1.49) is easily calculated

$$\Delta - T\frac{\partial\Delta}{\partial T} = -a - \frac{\pi}{2} . \quad (4.41)$$

With all the necessary parts calculated, the generalised Bohr-Sommerfeld quantisation condition finally reads

$$k_r(\pi - b_r - a) = 2\pi n , \quad (4.42)$$

where

$$k_r = k - 1 \quad , \quad b_r = \frac{kb - \frac{3\pi}{2}}{k-1} . \quad (4.43)$$

The form of the new quantisation condition is exactly the same as the first approximation of (4.20), only with a redefinition for the boundary and coupling constants. The shift in the coupling constant is to be expected. In the bulk case the first order corrections amount to a simple shift in k (4.12) as pointed by de Vega and Maillet [47]. The same holds for the half line case. The Bohr-Sommerfeld quantisation provides us with the original condition while the introduction of the next order term brings about the same shift $k \to k - 1$ as in the bulk.

In addition to k, the boundary constant has to be renormalised as well. It is not clear why this renormalisation is needed or whether it should appear at all. In a related paper examining the closely related a_{k-1} theory, Penati and Zanon [62] argued that renormalisation of boundary parameters has to be introduced in certain models to ensure integrability at the quantum level. The renormalisation of the coupling constant and its significance remains one of the open questions for the quantum CSG theory.

4.4 The Bootstrap Method

In the previous section we constructed the quantum spectrum using the semi-classical stationary-phase method. Although the results are not exact, they provide us with an accurate picture of the set of states. The same spectrum can be obtained using the completely different approach of the bootstrap method which was described in the introduction. It is based on the pioneering work of Cherednik [26], Ghoshal and Zamolodchickov [6], and Fring and Köberle [27]. This way we shall be able to compare complementary results to acquire an even more accurate spectrum. The idea behind this method is to construct the reflection factors through the boundary bootstrap relation of (1.62), and through the poles therein to identify boundary bound states. The process is analogous to the bulk case where the existence of bound states is indicated by poles found in the S-matrix.

Nevertheless there are quite a lot of drawbacks in this process. First of all in order to begin we need the quantum reflection factor for the particle of the theory. This cannot be obtained through any consistent procedure. Although the principles of analyticity, unitarity and crossing symmetry along with the boundary Yang-Baxter equation restrict the form of the reflection factor enough, it is impossible to pin it down completely. It is therefore a matter of "selecting" the correct reflection factor which should satisfy all constraints whilst introducing a pole corresponding to a boundary bound state.

After carefully deciding on the correct reflection factor for the particle, we are faced with a second problem. Although we are only concerned with poles in the reflection factor which appear on the physical strip (which in the half line case is $\Im(\theta) \,\epsilon\, [0, \pi/2]$), not all of them correspond to boundary bound states. It is quite difficult to explain the appearance of all the poles which lie within the physical strip and no work until now exists to offer a systematic treatment of poles encountered.

Another difficulty arises with the determination of the CDD factors in the reflection matrix. The restrictions imposed on the reflection factor may determine it up to a set of factors, quite like the CDD factors of the S-matrix. However the reflection CDD factors which are also restricted by the same constraints, introduce more poles which are also difficult to explain.

Considering all the above we shall attempt to construct the quantum boundary-bound state spectrum for the CSG theory but our study will be superficial. It is not our objective to fully explain the quantum structure of the model but rather to verify crudely our semi-classical results. The detailed explanation of poles in the reflection factors, the determination of a general form for the reflection CDD factors and the comparison of the results with similar theories are quite fascinating problems but are beyond the scope of this thesis. In the following section we shall present a suitable form for the quantum reflection factor for the particle of the CSG model.

4.4.1 Quantum reflection factor for the CSG particle

We begin our attempt to introduce a suitable reflection matrix K_1 for the CSG particle with the assumption that it should be made out of F factors that were defined in (4.14). It is a natural selection as both the boundary Yang-Baxter equation and the bootstrap equations relate reflection matrices with the S-matrix which is expressed only in terms of such functions. There is another advantage to this choice. Unitarity, real analyticity and $2\pi i$-periodicity requirements are automatically satisfied if K appears as a product of such factors. Some of the most important properties enjoyed by these functions include the following

$$F_Q(\theta)F_Q(-\theta) = 1 \ , \quad F_{2Q}(2\theta) = -F_Q(\theta)F_{Q+k}(\theta) \ , \quad F_Q(\theta + i\pi) = -F_{Q+k}(\theta) \ .$$

The first of the above is responsible for the fulfillment of the unitarity requirement. The remaining demonstrate basic transformations between rapidity and charge and will be used in the bootstrap and crossing symmetry equations.

It should be noted that since there is no degeneracy in the spectrum we expect K to be diagonal. This, in conjunction with a diagonal S-matrix, renders the boundary Yang-Baxter equation (1.60) trivial. In a previous section concerning the bulk case, the CSG particle was identified with the soliton $Q = 1$. In this context the first real constraint for K comes from the crossing symmetry relation (1.59), which for our case is written as

$$K_1(\theta)K_{\bar{1}}(\theta + i\pi) = S_{1,1}(2\theta) \ , \tag{4.44}$$

where $K_{\bar{1}}$ is the reflection of the antiparticle and $S_{1,1}$ is the two particle scattering matrix. Dorey and Hollowood noted that the CSG S-matrix is identical to the minimal a_{k-1} S-matrix which in turn can be recovered from the $a_{k-1}^{(1)}$ Affine Toda field theory (ATFT) when the parts involving the coupling constant are omitted. It is therefore reasonable to build our reflection matrix based on the proposed form for the particle reflection matrix of the boundary $a_{k-1}^{(1)}$ Affine Toda theory. In their paper Delius and Gandenberger [33] present a general form for the particle reflection matrix of the $a_{k-1}^{(1)}$ ATFT. As in the S-matrix case the reflection matrix is a product of two parts, out of which only one depends on the coupling constant. Each part satisfies the bootstrap independently so we can recover a K-matrix for our model by simply ignoring the coupling dependent pieces. The block notation implies $(x) = F_x(\theta)$ and shall be used henceforth in parallel with $F_x(\theta)$. Ignoring the parts involving the coupling constant, the remaining factors

$$K_n = \sum_{c=1}^{n}(c-1)(c-k) \quad , \quad n = 1..(k-1) \ , \tag{4.45}$$

constitute a complete set satisfying the crossing-symmetry condition (1.59) as well as the reflection bootstrap equation (1.60). This is not unexpected as both theories share the same minimal S-matrix . This however creates a problem as K does not contain any poles which can be related to boundary-bound states. This means that should any additional factors be added by hand to introduce the required poles, they should be added in such a way that they cancel between them in the crossing and bootstrap relations. This in turn suggests that the new factors are nothing more than CDD factors for the reflection matrix. Placing the poles in the CDD factors simplifies the whole procedure of determining a suitable particle reflection matrix since they satisfy simpler relations.

Real analyticity and unitarity conditions once again prompt us to construct our CDD factors out of block functions, so that the former are satisfied automatically. In order for the crossing relation to be satisfied any block factor (x) should be accompanied by the charge conjugate factor $(k-x)$. Delius and Gandenberger showed for such a combination the bootstrap closes.

Now that we have a consistent way of adding factors we need to find where the poles should appear. We begin with the simple formation of the boundary bound state described in (Fig. 1.12). During the process both energy and charge are conserved. We begin with the charge conservation. Far away from the boundary the soliton (particle) behaves as in the theory in the bulk. Its charge is equal to the normal soliton charge $Q_1 = \frac{k}{\pi}(\frac{\pi}{2} - a_1)$. After the formation of the bound state the charge Q_2 is given by the formula (4.19). Equating these yields

$$(\frac{\pi}{2} - a_1) = \frac{1}{2}(\pi - b - a_2) \ . \tag{4.46}$$

Using the same arguments we can write down an equation describing the conservation of energy

$$4m\cos(a_1)\cosh(\theta) - 2\cos(b) = 2m\cos(a_2) \ , \tag{4.47}$$

where the $-2\cos(b)$ term is the boundary energy contribution when the field is zero. From (4.46) and (4.47) we can determine the rapidity θ at which the boundary bound state is formed

$$\theta_n^{(0,n)} = i\frac{\pi}{k}\,(n+B) \quad \text{where} \quad B = \frac{k}{\pi}\,b - \frac{k}{2} \ . \tag{4.48}$$

For the above relation the quantisation condition of (4.20) was also used. Now that we have determined where the pole should be, we need to express it in block notation. It is easy to see that since (x) has a pole in the denominator at $\theta = i\frac{\pi}{k}x$, we therefore need a block $(n+B)$ and its counterpart $(k-n-B)$ for the CDD factor of K_n.

Nevertheless the blocks containing the pole should not be the only components of the CDD factor. On the contrary we expect the number of CDD factors in K_n to increase as n increases towards $\frac{k}{2}$ and then to decrease as it approaches $k-1$. Since we have $k-1$ particles and the theory is Z_k symmetric we expect $K_k = K_1$. Moreover since the reflection matrix remains the same under charge conjugation we expect $K_n = K_{k-n}$.

Although it is not easy to derive the general formula for the CDD factors in the K_n reflection matrix, we can decide on a minimal choice for K_1. We opt for a CDD factor containing only the first pole and its charge conjugate counterpart. The full reflection factor then reads

$$K_1^{(0)} = (1+B)(k-1-B)(1-k) \ . \tag{4.49}$$

The superscript denotes the boundary excitation state. In this particular case $K_1^{(0)}$ describes the reflection of a soliton of charge $Q = 1$ from an unexcited boundary.

$$\begin{tikzcd}
K_1^{(0)} \arrow[r, "\text{Reflection}", "\text{Bootstrap}"'] \arrow[d, "\text{Boundary Bootstrap}"'] & K_2^{(0)} \cdots\cdots K_{k-2}^{(0)} \arrow[r] \arrow[d] & K_{k-1}^{(0)} \arrow[d] \\
K_1^{(1)} \arrow[r] \arrow[d] & K_2^{(1)} \cdots\cdots K_{k-2}^{(1)} \arrow[r] \arrow[d] & K_{k-1}^{(1)} \arrow[d] \\
K_1^{(2)} \arrow[r] & K_2^{(2)} \cdots\cdots K_{k-2}^{(2)} \arrow[r] & K_{k-1}^{(2)} \\
\vdots & \vdots \qquad \vdots & \vdots \\
K_1^{(k-1)} \arrow[r] & K_2^{(k-1)} \cdots\cdots K_{k-2}^{(k-1)} \arrow[r] & K_{k-1}^{(k-1)}
\end{tikzcd}$$

Figure 4.1: The Bootstrap Programme

We ignore the poles coming from the non-CDD part of $K_1^{(0)}$. We shall address this issue at the closing stages of this chapter. The pole associated with the $(1+B)$ factor corresponds to the lowest boundary bound state with energy

$$M_b^{(1)} = \frac{km}{2\pi}\left|\cos(\pi - b - \frac{2\pi}{k})\right| . \tag{4.50}$$

which is formed when a soliton of charge $Q = 1$ fuses with the boundary. The conjugate term $(k-1-B)$ also has a pole but does not lie within the physical strip $\Im(\theta) \, \epsilon \, (0, \frac{\pi}{2})$.

With $K_1^{(0)}$ as a starting point we can construct the whole set of $K_n^{(0)}$ factors by using the reflection bootstrap (1.60). In addition, we can apply the boundary bootstrap (1.62) to each of them to obtain the corresponding reflection matrices from the excited boundary. In both cases new block factors are generated carrying poles which indicate new bound states. If our choice for the CDD factor in (4.49) is correct we expect the bootstrap to close, i.e. to end up with a finite spectrum of states which is repeated after k steps. Pictorially this can be seen in (Fig. 4.1)

As a starting point we can calculated the reflection factor of a charge $Q = 2$ soliton bouncing off the unexcited boundary

$$K_2^{(0)}(\theta) = K_1^{(0)}(\theta - \frac{i\pi}{k})K_1^{(0)}(\theta + \frac{i\pi}{k})S_{1,1}(2\theta). \tag{4.51}$$

Substituting $K_1^{(0)}$ from (4.49) and $S_{1,1}$ from (4.15) we finally get

$$K_2^{(0)} = (2+B)(k-2-B)(B)(k-B)(1-k)(1)(2-k) . \tag{4.52}$$

This result confirms that the general formula of (4.45) forms a complete set. Henceforth we shall ignore the basic blocks of the reflection matrix as they are a consistent set and concentrate only on the CDD factors which carry all necessary information about boundary bound states. The first four terms are the CDD factor for $K_2^{(0)}$ which introduce two new poles. An equally straightforward calculation is that of $K_4^{(0)}$ which is found to be

$$K_4^{(0)} = \frac{(B+4)(B+2)(k-B+4)(k-B+2)}{(-B)(-B+2)(-k+B+2)(-k+B+4)} \ . \tag{4.53}$$

A pattern begins to emerge from these simple results. The CDD factor of $K_n^{(0)}$ always involves the block pair $(n+B)(k-n-B)$. This is the first consistent appearance of poles in the reflection matrices. This is also the complete set of poles that are produced by the bootstrap programme. With an arbitrary coupling constant k, it is not easy to demonstrate the complete mechanism of pole production. this can easily be seen by fixing k to an integer value. One can then see the whole range of poles produced and how the bootstrap miraculously closes.

4.4.2 The boundary bootstrap

In the previous section we proposed a suitable expression for the $K_1^{(0)}$ reflection factor and saw how through the reflection bootstrap all the $K_n^{(0)}$ factors can be obtained. We now turn to the boundary bootstrap to construct the boundary reflection factors of solitons bouncing off an excited boundary. We shall only examine the boundary bootstrap for the particle reflection factor. Through the reflection bootstrap all other states can be built (Fig. 4.1). Once again we are facing the problem that with k arbitrary and a charge-varying S-matrix we are not in a position to identify all poles or cancel terms in the reflection matrix. We therefore shall attempt to find a general formula for the energy difference between adjacent boundary bound states. This should be enough to build the whole spectrum of states beginning with the energy of the first bound state.

We begin with the boundary bootstrap equation (1.62) for $K_1^{(1)}$ which is

$$K_1^{(1)} = K_1^{(0)} S_{1,1}(\theta + \theta_1^{(0,1)}) S_{1,1}(\theta - \theta_1^{(0,1)}) \ , \tag{4.54}$$

where $\theta_1^{(0,1)} = i\frac{\pi}{k}(1+B)$ denotes the pole in $K_1^{(0)}$. The product of S-matrices in the above relation is equal to a shift in their charge. In general the following relation holds for any ψ

$$S_{1,1}(\theta+i\psi)S_{1,1}(\theta-i\psi) = (2+\frac{k}{\pi}\psi)(2-\frac{k}{\pi}\psi) \ . \tag{4.55}$$

Putting everything together yields

$$K_1^{(1)} = (1+B)(k-1-B)(3+B)(1-B) \ . \tag{4.56}$$

Once again we have only used the CDD factors and ignored terms not involving the boundary constant. We see through this procedure a new pole appears from the block $(3+B)$ corresponding to the absorption of a charge $Q=1$ particle into the charge $Q_b=1$ boundary. This will be used as an input to find a new pole in $K_1^{(2)}$

$$K_1^{(2)} = (k-1-B)(3+B)(1-B)(5+B) \ , \tag{4.57}$$

where a cancellation has already taken place. The new pole comes from the block factor $(5+B)$ which will be used in the next step. The expression for $K_1^{(3)}$ is

$$K_1^{(3)} = (k-1-B)(1-B)(5+B)(7+B) \ , \tag{4.58}$$

where new factor $(-3-B)$ has cancelled with the block factor $(3+B)$ and the factor $(7+B)$ has introduced a new pole. This procedure will not continue indefinitely. After k steps we expect to return to original form of $K_1^{(0)}$. However it is clear that the $K_1^{(n)}$ reflection matrix will have a pole indicated by the block factor $(2n+1+B)$ at $\psi_n = \frac{\pi}{k}(2n+1+B)$. Having a general formula about the n-th pole allows us to write down a recursive relation for the energies of the bound states. We begin with the energy of the first bound state. The difference between the first excited boundary state and the non-excited boundary is

$$E_1 - E_0 = A\cos(\psi_0) \ . \tag{4.59}$$

The right hand side is equal to the mass of the incoming particle that binds with the boundary at a fixed angle $\theta = i\psi_0$. The parameter A is related to the mass of the particle and will be determined from the comparison with the semi-classical results. The formula may be used recursively to finally yield

$$E_{n+1} - E_n = A\cos(\psi_n) = A\cos(\frac{\pi}{k}(2n+1+B)) \ . \tag{4.60}$$

This formula should be compared with the one derived from the semi-classical approach (4.24). Both describe the exact same energy gaps between two bound states. The arbitrary parameter in (4.60) can be read directly from (4.24)

$$A = \frac{km}{\pi} \cos\left(\frac{\pi}{2} - \frac{\pi}{k}\right) . \tag{4.61}$$

This is not surprising. The energy difference between bound state $A\cos(\psi_n)$ is equal to the energy of a soliton

$$M_s = \frac{mk}{\pi} \cos(a)\cosh(\theta) . \tag{4.62}$$

The parameter A is exactly the charge of the incoming soliton. As k becomes large the charge of the soliton approaches zero and the soliton becomes a particle. In the limit that $k \to \infty$ (the classical limit) the soliton becomes infinitesimally small and the boundary-bound states spectrum become continuous.

We conclude this section with a brief discussion about the poles appearing in the reflection matrix. We expect the non-CDD poles in (4.45) to be explained in terms of a on-shell triangle diagram as is the case for the $a_n^{(1)}$ affine Toda theory. A full discussion can be found in [33]. The remaining poles appear in the CDD factors and are the ones associated with the boundary-bound states. The full set of poles generated from the bootstrap programme come from the general blocks $(n+B)$ and $(k-n-B)$ which have poles at

$$\psi_n = \frac{\pi}{k}n - \frac{\pi}{2} + b \quad \text{and} \quad \psi_n' = \pi - \left(\frac{\pi}{k}n - \frac{\pi}{2} + b\right) \tag{4.63}$$

From the forms above we can see that if θ_n lies in the physical strip $(0, \frac{\pi}{2})$ then θ_n' does not, and vice versa. Assuming that the pole is at θ_n then it lies in the physical strip if

$$\frac{\pi}{2} > \psi_n > 0 \quad \Leftrightarrow \quad 1 > \frac{n}{k} + \frac{b}{\pi} > \frac{1}{2} \tag{4.64}$$

As soon the above condition is no longer true then the pole appears in the conjugate block at ψ_n'. This however does not alter any of our results. From (4.60) we can see that ψ_n or ψ_n' appear in the argument of the cosine so both correspond to the same absolute energy difference between two states.

4.5 Discussion

The fact that the results of the semi-classical method and those of the bootstrap method coincide is remarkable. Although it may seem that the poles in the bootstrap method were "manufactured" using the ones from the semi-classical approach, this is not so. The introduction of the pole in $K_1^{(0)}$ which came from the semi-classical spectrum is not restrictive. The bootstrap approach reproduces exactly the same energy gaps between quantum states as the semi-classical approach. This can be seen by direct comparison of the relations (4.24) and (4.60). However in the bootstrap approach the energy of the lowest state, which is indicated by the pole in the particle reflection factor, is not defined and may be chosen arbitrary. At this point we choose the lowest energy of the semi-classical result as a starting point and the rest of the quantum spectrum is obtained through the recursive relation of (4.60). Any general pair of CDD blocks will have produced the exact same results and through the comparison with the semi-classical results the original choice will have emerged. The reflection and boundary bootstrap both close because the minimal choice for K_n (4.45) is consistent with our S-matrix and because the introduced blocks are CDD factors.

One final point to be made is that although we have used for our comparison equations (4.20) and (4.24) which correspond to the first approximation in the semi-classical approach, we can extend our results to agree with the first order corrections by simply substituting everywhere $k = k_r$ and $b = b_r$. The redefinition of both parameters change nothing in the bootstrap approach which yields the exact same results.

Chapter 5

Conclusion

Integrable field theories are a fascinating subject with plenty of applications in the physical world. One such theory is the complex sine-Gordon model, a generalisation of the sine-Gordon theory with a global $U(1)$ degree of freedom. Although the model was introduced almost three decades ago, it failed to attract enough attention in the research world. Even though some aspects of the model have been studied, it is only recently that it has been consistently explored within the framework of homogeneous sine-Gordon theories.

In this thesis we have studied a different aspect of the CSG model. We have introduced a boundary and considered the theory on a half line. Our goal was to find specific boundary conditions that would be consistent with integrability and through which we could explore the interaction of solutions with the boundary and the existence of boundary-bound states.

Our investigation has forced us to readdress a number of inconsistencies arising in the bulk case from previous treatments. We have shown that soliton and antisoliton solutions do not belong into distinct classes but are in fact the same entity. In the absence of topology there exists no way to separate such solutions. Only in the chargeless limit of the theory where the sine-Gordon model is recovered solitons regain their topological character and such a classification may hold. Another issue that needed clarification was the existence of breather solutions. We have proposed a elegant way to create breather solutions by analytic continuation that satisfy the equations of motion. An amazing property of breather solutions was also discovered. Since solitons and breathers are not topological objects, then it is possible to fix the parameters in the breather solutions so that the breather collapses to a single soliton solution.

Following that we have used the zero curvature condition to construct the infinite number of conserved currents. With the introduction of a boundary, we demanded that all parity even quantities are conserved and through this restriction we found the form of the integrable boundary condition. This has provided us with the form of the boundary term in the CSG Lagrangian, thus allowing us to reexamine the vacuum of the theory on a half line. We have investigated the circumstances under which the two soliton solution satisfies the boundary condition and through the method of images have calculated the time delay induced when a soliton reflects off the boundary wall. Finally we have addressed the issue of boundary-bound states, examining the static soliton and boundary breather case.

Continuing our investigation we have examined the quantum CSG theory on a half line. The quantum CSG in the bulk has already been studied in a number of papers and the results were used in the half line case. We adopted the semi-classical stationary-phase method to obtain the boundary-bound spectrum of the CSG model. The first order corrections produced the same finite renormalisation of the coupling constant as in the bulk case, whilst the boundary constant underwent a similar redefinition. In addition to the semi-classical approach, we have used the bootstrap method to calculate the energy difference between boundary-bound states. A minimal choice for the reflection factor of the CSG particle was proposed based on the corresponding factor of the a_{k-1} theory, whilst new factors were introduced bearing the necessary poles for our boundary-bound states spectrum. In order to avoid any complication with the constraint equations, the poles were placed in the CDD factors simplifying the whole process. Using the boundary bootstrap equation a recursive formula was created to obtain the energy levels of states. A comparison showed both methods yielding the same results thus confirming their validity.

We end our discussion with a few open questions and opportunities for further research. We begin with the existence of breathers. In this thesis we found a way to create breathers by analytic continuation of the two soliton solution when the parameters have been fixed in a specific way. Nevertheless, we still lack direct confirmation through the equations of motion. This also raises the question whether a valid solution is obtained without fixing any of the parameters. It is our belief that analytic continuation will not in general produce a solution, but that other solutions which describe the 'fusion' of two unequally charged soliton should exist.

Another point which needs further exploration is the full set of boundary conditions that are consistent with integrability. Apart from the solution presented and a set of isolated Dirichlet-like boundary conditions, one can imagine more complicated forms involving higher derivatives or even time-like boundary conditions. Such solutions would provide an extremely interesting topic of research.

Furthermore, the boundary quantum CSG model is not yet fully explored. Continuing with the semi-classical quantisation it would be quite interesting to see the next level of quantum corrections that need to be introduced and to what extend these will agree with the ones from the bulk case. In addition, the renormalisation of the boundary constant may or may not continue beyond this order and the reason for its existence needs further investigation.

One could also contemplate testing any of our conjectures for the reflecting K using perturbation theory [63].

Last but not least, the connexion of the theory with similar models is of great importance. As we have already mentioned the model is the simplest of the homogeneous sine-Gordon theories and it would be interesting to see how the results of the boundary CSG translate for the other members of this group. The close connexion with a_{k-1} which was established through the S-matrix and the reflection matrix present the best indication that such connexion exists and that results are expected to be related. Finally, a thorough investigation of the pole structure of the boundary CSG model would provide a clearer picture of the quantum case which was not implemented in this treatment.

Appendix A

The theory of optical pulses

One of the applications of the complex sine-Gordon theory is to provide a field theory description for optical pulses propagating within a non-linear medium. In this section we present the theory of optical pulses and its connexion to the CSG model. Moreover, we consider the boundary problem and demonstrate integrability once again by building the infinite number of conserved quantities as we did in the third chapter. However there is a subtle difference. In order to achieve a realistic description a different gauge has to be chosen. This complicates our effort to construct the conserved quantities and to find boundary conditions which preserve integrability.

A.1 Introduction

Non-linear optical systems lack a satisfactory field theory description. It is their non-linear character which makes the usual description through the scalar potential ϕ and vector potential A of electromagnetism unfeasible. Especially the vector potential, looses its meaning through the non-linear interactions with matter. Therefore a Lagrangian formulation is impossible due to the absence of a potential like variable. The only way to describe such systems is to adopt a semi-classical approach and use the electric field itself as a field variable. By using special functions for the latter, the Maxwell-Bloch equation -which describes electromagnetic interactions- is satisfied and so a description of non-linear systems is possible.

The only description which introduces a Lagrangian formulation and in which a potential-like variable is used, is due to McCall and Hahn [48]. The system is described, in certain conditions, by a potential variable $\phi(x,t)$, which is shown to satisfy the sine-Gordon equation. In this way the self-induced transparency effect (SIT), the phenomenon where an optical pulse in a non-linear medium propagates without significant energy loss, can be interpreted in terms of the soliton solutions that the sine-Gordon model possesses.

The work of Park and Shin [49, 64, 65] takes the previous approach a step further. They argue that a better choice would be the complex sine-Gordon model substituting the potential variable by the matrix potential g. In comparison with its predecessor, the model seems to have great advantages: frequency detuning, modulation and inhomogeneous broadening effects can all be described within the framework of the complex sine-Gordon theory, while the restriction to two level non-degenerate systems holds no more. In their papers, Shin and Park rewrite the Maxwell-Bloch equations

$$\begin{aligned} (\partial_1^2 - \frac{n^2}{c^2}\partial_0^2)\mathbf{E} &= \frac{4\pi}{c^2}\partial_0^2 \int dv \mathrm{tr}\rho\ \mathbf{d} && \text{Maxwell equation} \\ i\hbar(\partial_0 + v\partial_1)\rho &= [(H_0 - \mathbf{E}\ \ \mathbf{d}), \rho] && \text{Bloch equation} \end{aligned}$$

in a dimensionless form:

$$\begin{aligned} \bar{\partial}E\ + 2\ \beta P &= 0\ , \\ \partial D - E^* P - E P^* &= 0\ , \\ \partial P + 2i\xi P + 2ED &= 0\ . \end{aligned} \tag{A.1}$$

In this form, E represents the electric field, D is the population inversion and P is the polarization. The parameter ξ should not be confused with the coupling constant of the previous chapters. In order to produce the sine-Gordon equation, a potential-like variable is introduced through the relation

$$\phi(x,t) \equiv \int_{-\infty}^{t} E\ dt'\ .$$

By identifying $E = E^* = \partial\phi$, $P = -\sin(2\phi)$ and $D = \cos(2\phi)$ the sine-Gordon equation arises

$$\bar{\partial}\partial\phi = 2\beta\sin(2\phi)\ .$$

In order for the complex sine-Gordon to emerge, we parameterize the three quantities E, D, and P with three scalar fields ϕ, θ and η

$$D = \cos(2\phi), \tag{A.2}$$

$$P = ie^{i(\theta-2\eta)}\sin(2\phi)\ , \tag{A.3}$$

$$E = e^{i(\theta-2\eta)}(2\partial\eta\cot(\phi) - i\partial\phi)\ , \tag{A.4}$$

When the above expressions are inserted into (A.2), we obtain a set of second order differential equations

$$\begin{aligned} \bar{\partial}\partial\phi + 4\frac{\cos(\phi)}{\sin^3(\phi)}\partial\eta\bar{\partial}\eta - 2\beta\sin(2\phi) &= 0\ , \\ \bar{\partial}\partial\eta - \frac{2}{\sin(2\phi)}(\bar{\partial}\eta\partial\phi + \partial\eta\bar{\partial}\phi) &= 0\ , \end{aligned} \tag{A.5}$$

and a couple of first order constraint equations

$$\begin{aligned} 2\cos^2(\phi)\partial\eta - \sin^2\phi\partial\theta - 2\xi\sin^2\phi &= 0\,, \\ 2\cos^2\phi\bar{\partial}\eta + \sin^2\bar{\partial}\theta &= 0\,. \end{aligned} \tag{A.6}$$

Having expressed the Maxwell-Bloch theory as the CSG equations, we can use the matrix potential formalism of the second chapter. The $SU(2)$ matrix potential g of (2.24) is now defined through the relation

$$\begin{pmatrix} D & P \\ P^* & -D \end{pmatrix} = g^{-1}\sigma_3 g \tag{A.7}$$

The Bloch equation emerges from the identity relation

$$\partial(g^{-1}\sigma_3 g) = \left[g^{-1}\sigma_3 g, g^{-1}\partial g\right]\,, \tag{A.8}$$

and the identification

$$g^{-1}\partial g + R = \begin{pmatrix} i\xi & -E \\ E^* & -i\xi \end{pmatrix}\,. \tag{A.9}$$

where R is an anti-Hermitian matrix which commutes with $g^{-1}\sigma_3 g$. The Maxwell equation can also be expressed in terms of g

$$\bar{\partial}\left(g^{-1}\partial g + R\right) = \begin{pmatrix} 0 & -\bar{\partial}E \\ \bar{\partial}E^* & 0 \end{pmatrix} = \beta\left[\sigma_3, g^{-1}\sigma_3 g\right]\,. \tag{A.10}$$

The variation of the gauged WZW action (2.18) produces the Maxwell-Bloch equations when the gauge is fixed as follows

$$W = i\xi\sigma_3 \quad , \quad \bar{W} = 0\,, \tag{A.11}$$

and which in a zero curvature form is

$$\left[\partial + \begin{pmatrix} i\beta\lambda + i\xi & -E \\ E^* & -i\beta\lambda - i\xi \end{pmatrix}, \bar{\partial} - \frac{i}{\lambda}\begin{pmatrix} D & P \\ P^* & -D \end{pmatrix}\right] = 0\,. \tag{A.12}$$

A.2 Conserved currents

In this section we use the same method as in the third chapter to construct the infinite set of conserved currents and through them to obtain boundary conditions consistent with integrability. However, the gauge fixing of (A.11) is different to our choice in the second chapter (2.27). Although this shall modify most of the calculations, our original results can be recovered in the limit $\xi = 0$. We follow the exact same steps for the abelianisation of the Lax pair noting that only the definition of the parameter Λ has changed

$$\Lambda = \beta\lambda + \xi \ . \tag{A.13}$$

The expansion of the conserved quantities J and $\bar{J}$ of (3.9) and (3.12) respectively, now yield

$$\begin{aligned} J &= -i\lambda\beta - i\xi - \frac{i}{2\beta}EE^*\left(\frac{1}{\lambda}\right) + \left(\frac{i\xi}{2\beta^2}EE^2 - \frac{1}{8\beta^2}(\partial E^* E - \partial E E^*)\right)\left(\frac{1}{\lambda^2}\right) , \\ \bar{J} &= \frac{iD}{\lambda} + \frac{1}{4\beta}(E^* P - E P^*)\left(\frac{1}{\lambda^2}\right) . \end{aligned}$$

These expressions should be compared with the those of (3.15). As we can see the parameter ξ has modified our previous results significantly by introducing new terms.

In the third chapter we argued that only parity even quantities are preserved. The conserved currents of (3.15) were neither parity even or odd and a "reflected" set had to be constructed that would combine with the original set to produce pure parity states. Since the system was parity invariant the new set was easily obtained by substituting $\partial \to \bar{\partial}$. This is not true in this case. Although the equations of motion are parity invariant, the first order constraints because of the gauge fixing of (A.11) are not. The new underlying symmetry of the system has first to be identified before we construct the set of "reflecting" currents. To this end we shall try to build a new Lax pair which should have the same form as the original one (A.12) but with the derivatives interchanged and the field variables E,P and D slightly altered. We begin with a gauge transformation for $\bar{A}$

$$\bar{A} \to \tilde{\bar{A}} = g\bar{A}g^{-1} + \bar{\partial}gg^{-1} = \frac{i\sigma_3}{\lambda} + \begin{pmatrix} 0 & \tilde{E} \\ -\tilde{E}^* & 0 \end{pmatrix} , \tag{A.14}$$

where

$$g = \begin{pmatrix} e^{2i\eta}cos\phi & isin\phi e^{i\theta} \\ isin\phi e^{-i\theta} & e^{-2i\eta}cos\phi \end{pmatrix} , \tag{A.15}$$

and

$$\tilde{E} = E \left| \begin{array}{c} \eta \to -\eta \\ \phi \to -\phi \\ \partial \to \bar{\partial} \end{array} \right| . \tag{A.16}$$

The same gauge transformation on A yields

$$A \to \tilde{A} = gAg^{-1} + \partial g g^{-1} = g \begin{pmatrix} -i\xi & E \\ -E^* & i\xi \end{pmatrix} g^{-1} - i(\beta\lambda) g\sigma_3 g^{-1} + \partial g g^{-1} . \tag{A.17}$$

We can simplify the above expression using

$$g \begin{pmatrix} -i\xi & E \\ -E^* & i\xi \end{pmatrix} g^{-1} + \partial g g^{-1} = -i\xi\sigma_3 . \tag{A.18}$$

It is easy to show that the following relation is true:

$$ig\sigma_3 g^{-1} = i \begin{pmatrix} \tilde{D} & \tilde{P} \\ \tilde{P}^* & -\tilde{D} \end{pmatrix} , \tag{A.19}$$

where

$$\tilde{D}, \tilde{P}, \tilde{P}^* = D, P, P^* \left| \begin{array}{c} \eta \to -\eta \\ \phi \to -\phi \end{array} \right| . \tag{A.20}$$

The form of $\tilde{A}$ has now simplified to

$$\tilde{A} = -i\beta\lambda \begin{pmatrix} \tilde{D} & \tilde{P} \\ \tilde{P}^* & -\tilde{D} \end{pmatrix} - i\xi\sigma_3 . \tag{A.21}$$

Another gauge g' transformation has to be introduced to complete the switch in the Lax pair

$$g' = \exp(\frac{i\xi\sigma_3(t-x)}{2}) \tag{A.22}$$

As a result a phase factor will appear in front of $\tilde{E}$ and $\tilde{P}$ and their complex conjugates

$$\tilde{P} \to \tilde{P}\ e^{i\xi t - x)} \quad , \quad \tilde{P}^* \to \tilde{P}^*\ e^{-i\xi(t-x)} , \tag{A.23}$$

$$\tilde{E} \to \tilde{E}\ e^{i\xi(t-x)} \quad , \quad \tilde{E}^* \to \tilde{E}^*\ e^{-i\xi(t-x)} . \tag{A.24}$$

The extra factors can be absorbed if we let

$$\theta \to \theta + i\xi(t - x) \ . \tag{A.25}$$

Using the following set of equations

$$\partial g g^{-1} = i\xi\sigma_3 \quad , \quad \bar{\partial} g g^{-1} = 0 \ , \tag{A.26}$$

we get the following expressions for $\tilde{A}$ and $\tilde{\bar{A}}$

$$A \ = \ \frac{i}{\lambda^*}\begin{pmatrix} \tilde{D} & \tilde{P} \\ \tilde{P}^* & -\tilde{D} \end{pmatrix} , \tag{A.27}$$

$$\tilde{\bar{A}} \ = \ -i\sigma_3(\beta\lambda^*) + \begin{pmatrix} 0 & \tilde{E} \\ -\tilde{E}^* & 0 \end{pmatrix} , \tag{A.28}$$

where $-\beta\lambda = \lambda^{*-1}$. The final step is to change the factor $\beta\lambda^*$ in $\tilde{\bar{A}}$ to the desired form of $\beta\lambda^* + \xi$. This is easily accomplished by yet another gauge transformation g''

$$g'' = \exp(-i\sigma_3\xi\frac{(t-x)}{2}) \ . \tag{A.29}$$

Once again the difference that appears can be absorbed in θ. The new form of the Lax pair is

$$\left[\bar{\partial} + \begin{pmatrix} i\beta\lambda^* + i\xi & -\tilde{E} \\ \tilde{E}^* & -i\beta\lambda^* - i\xi \end{pmatrix}, \partial - \frac{i}{\lambda^*}\begin{pmatrix} \tilde{D} & \tilde{P} \\ \tilde{P}^* & -\tilde{D} \end{pmatrix}\right] = 0 \tag{A.30}$$

Comparing this with the Lax pair of (A.12) we can see that it has the exact same form, with the fields E , P and D now replaced with $\tilde{E}$, $\tilde{P}$, and $\tilde{D}$, and the derivatives having switched places. The new Lax pair reproduces the same equations of motion for the new field variables but with the partial derivatives interchanged. Through this process we have managed to extract the hidden symmetry of the system. This is represented in the following transformations which leave both the equations of motion and the constraint equations unchanged

$$\begin{aligned} \theta &\to \theta - 2\xi x \\ \eta &\to -\eta \\ \phi &\to -\phi \\ \partial &\to \bar{\partial} \\ \lambda^* &\to -\tfrac{1}{\beta\lambda} \end{aligned} \tag{A.31}$$

We are now in a position to write down pure parity combinations of the conserved currents.

A.2.1 Conserved Currents

For the construction of the conserved quantities we follow the same procedure as in section (3.1.1). The first term of the expansion of (3.18) is given by

$$\partial_0\left(EE^*+\tilde{E}\tilde{E}^*+2\beta(D+\tilde{D})\right)=\partial_1\left(\tilde{E}\tilde{E}^*-EE^*+2\beta(D-\tilde{D})\right), \qquad (A.32)$$

where now $\tilde{E}$,$\tilde{P}$ and $\tilde{D}$ are given by (A.16) and (A.20). By direct substitution we obtain the following relation

$$\partial_0\left[2\tfrac{cos^2\phi}{sin^2\phi}[(\partial_0\eta)^2+(\partial_1\eta)^2]+\tfrac{1}{2}(\partial_0\phi)^2+\tfrac{1}{2}(\partial_1\phi)^2+\beta cos2\phi\right]=$$
$$\partial_1\left[4\tfrac{cos^2\phi}{sin^2\phi}\partial_0\eta\partial_1\eta+\partial_0\phi\partial_1\phi\right] \qquad (A.33)$$

Integration with respect to x over the semi-infinite interval yields

$$\int_{-\infty}^{0}\partial_1\left(4\frac{cos^2\phi}{sin^2\phi}\partial_0\eta\partial_1\eta+\partial_0\phi\partial_1\phi\right)dx=\left[4\frac{cos^2\phi}{sin^2\phi}\partial_0\eta\partial_1\eta+\partial_0\phi\partial_1\phi\right]_{-\infty}^{0} \qquad (A.34)$$

At infinity the fields are zero, so the left hand side which is evaluated at the boundary $x=0$. This can be expressed as a total time derivative when suitable boundary conditions for the fields are chosen. Let this be a total derivative of a function W, depending on the fields

$$\frac{dW}{dt}=\frac{\partial W}{\partial\phi}\partial_0\phi+\frac{\partial W}{\partial\eta}\partial_0\eta\ . \qquad (A.35)$$

It is very easy to make the following identification

$$\tfrac{\partial W}{\partial\phi}=f(\phi,\eta)\quad \tfrac{\partial W}{\partial\eta}=4\tfrac{cos^2\phi}{sin^2\phi}g(\phi,\eta) \qquad (A.36)$$

where the boundary conditions of the two fields take the form

$$\partial_1\phi=f(\phi,\eta)\quad ,\quad \partial_1\eta=g(\phi,\eta) \qquad (A.37)$$

Since no restrictions are imposed to the choice of f and g, one has to examine the next conserved quantity which is associated with the $\frac{1}{\lambda^2}$ term and is derived from the addition of

$$\partial_0(2i\xi EE^*+\frac{1}{2}(\partial EE^*-\partial E^*E)-8i\beta\partial\eta cos^2\phi)=$$
$$\partial_1(-2i\xi EE^*-\frac{1}{2}(\partial EE^*-\partial E^*E)-8i\beta\partial\eta cos^2\phi)\ , \qquad (A.38)$$

and

$$\partial_0(2i\xi\tilde{E}\tilde{E}^* + \frac{1}{2}(\bar{\partial}\tilde{E}\tilde{E}^* - \bar{\partial}\tilde{E}^*\tilde{E}) - (\tilde{E}^*\tilde{P} - \tilde{E}\tilde{P}^*) =$$
$$\partial_1(2i\xi\tilde{E}\tilde{E}^* + \frac{1}{2}(\bar{\partial}\tilde{E}\tilde{E}^* - \bar{\partial}\tilde{E}^*\tilde{E}) + (\tilde{E}^*\tilde{P} - \tilde{E}\tilde{P}^*) \tag{A.39}$$

Substituting in the explicit expressions for the fields we finally get the same boundary conditions as before

$$\partial_1\eta = 0 \quad \partial_1\phi = C_1\cos\phi \ . \tag{A.40}$$

This suggests that the choice of gauge does not destroy parity invariance, but rather disguises it in a hidden "parity-like" symmetry. Through this symmetry we are once again able to construct pure-parity conserved quantities, and derive the exact same boundary conditions.

Appendix B

The two-soliton solution

Although the CSG two-soliton solution has a relatively simple form when expressed in terms of one-soliton solutions (2.49), the full expression in terms of t and x is rather cumbersome making it virtually impossible to show that it satisfies the equations of motion. As far as we are aware no explicit formula has ever appeared in the literature.

The CSG two-soliton solution is written as

$$u_{2s} = \frac{N}{D}\ , \tag{B.1}$$

where

$$\begin{aligned}
N \;=\;& -i\cos(a_2)\,{J_2}^2\left(-2\,e^{\theta_1}\sin(a_2)\,e^{\theta_2}+\left(e^{\theta_1}\right)^2\sin(a_1)+\left(e^{\theta_2}\right)^2\sin(a_1)\right)\\
&\times J_1 e^{im\sin(a_2)(\cosh(\theta_2)t+\sinh(\theta_2)x)}\cosh\left(m\cos(a_1)\left(\cosh(\theta_1)(x-x_1)+\sinh(\theta_1)\,t\right)\right)\\
&+i\cos(a_1)\,{J_1}^2\left(-\left(e^{\theta_1}\right)^2\sin(a_2)+2\,e^{\theta_2}\sin(a_1)\,e^{\theta_1}-\left(e^{\theta_2}\right)^2\sin(a_2)\right)\\
&\times J_2 e^{im\sin(a_1)(\cosh(\theta_1)t+\sinh(\theta_1)x)}\cosh\left(m\cos(a_2)\left(\cosh(\theta_2)(x-x_2)+\sinh(\theta_2)\,t\right)\right)\\
&+\left(e^{\theta_1}-e^{\theta_2}\right)\left(e^{\theta_1}+e^{\theta_2}\right)\cos(a_2)\cos(a_1)\,{J_1}^2 J_2 e^{im\sin(a_1)(\cosh(\theta_1)t+\sinh(\theta_1)x)}\\
&\times\sinh\left(m\cos(a_2)\left(\cosh(\theta_2)(x-x_2)+\sinh(\theta_2)\,t\right)\right)\\
&-\left(e^{\theta_1}-e^{\theta_2}\right)\left(e^{\theta_1}+e^{\theta_2}\right)\cos(a_2)\cos(a_1)\,{J_2}^2 J_1 e^{im\sin(a_2)(\cosh(\theta_2)t+\sinh(\theta_2)x)}\\
&\times\sinh\left(m\cos(a_1)\left(\cosh(\theta_1)(x-x_1)+\sinh(\theta_1)\,t\right)\right)\ ,
\end{aligned}$$

and

$$\begin{aligned}
D \;=\;& J_1J_2\left(-\left(e^{\theta_1}\right)^2+2\,e^{\theta_2}e^{\theta_1}\sin(a_1)\sin(a_2)-\left(e^{\theta_2}\right)^2\right)\\
&\times\cosh\left(m\cos(a_2)\left(\cosh(\theta_2)(x-x_2)+\sinh(\theta_2)\,t\right)\right)\\
&\times\cosh\left(m\cos(a_1)\left(\cosh(\theta_1)(x-x_1)+\sinh(\theta_1)\,t\right)\right)\\
&+\cos(a_1)\cos(a_2)\,{J_2}^2 e^{\theta_2+\theta_1-im(\sin(a_1)(\cosh(\theta_1)t+\sinh(\theta_1)x)-\sin(a_2)(\cosh(\theta_2)t+\sinh(\theta_2)x))}\\
&+\cos(a_2)\cos(a_1)\,{J_1}^2 e^{\theta_2+\theta_1+im(\sin(a_1)(\cosh(\theta_1)t+\sinh(\theta_1)x)-\sin(a_2)(\cosh(\theta_2)t+\sinh(\theta_2)x))}\\
&+2\,J_1J_2\cos(a_1)\sinh\left(m\cos(a_1)\left(\cosh(\theta_1)(x-x_1)+\sinh(\theta_1)\,t\right)\right)\cos(a_2)\\
&\times\sinh\left(m\cos(a_2)\left(\cosh(\theta_2)(x-x_2)+\sinh(\theta_2)\,t\right)\right)e^{\theta_2+\theta_1}\ .
\end{aligned}$$

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